

IV. Space-time symmetries

❖ Conservation laws have their origin in symmetries and invariance properties of the underlying interactions

☉ Exact symmetry implies a conservation law \Rightarrow an observable which *absolute* value can not be defined (“*non-observable*”)

Symmetries, conservation laws and “non-observables”:

Symmetry transformation	Conservation law or selection rule	Non-observable
Space translation: $\mathbf{x} \rightarrow \mathbf{x} + \delta\mathbf{x}$	momentum	absolute spatial position
Rotation: $\bar{\mathbf{x}} \rightarrow \bar{\mathbf{x}}'$	angular momentum	absolute spatial direction
Time translation: $t \rightarrow t + \delta t$	energy	absolute time
Reflection: $\bar{\mathbf{x}} \rightarrow -\bar{\mathbf{x}}$	parity	“handedness” (absolute generalized right/left)
Charge conjugation: $q \rightarrow -q$	particle-antiparticle symmetry	absolute sign of electric charge
$\psi \rightarrow e^{iq\theta}\psi$	charge q	relative phase between states of different q
$\psi \rightarrow e^{iL\theta}\psi$	lepton number L	relative phase between states of different L
$\psi \rightarrow e^{iB\theta}\psi$	baryon number B	relative phase between states of different B

Translational invariance

- ❖ When a closed system of particles is moved from one position in space to another, its physical properties do not change

Considering an infinitesimal translation $\vec{x}_i \rightarrow \vec{x}'_i = \vec{x}_i + \delta\vec{x}$, the Hamiltonian of the system transforms as:

$$H(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) \rightarrow H(\vec{x}_1 + \delta\vec{x}, \vec{x}_2 + \delta\vec{x}, \dots, \vec{x}_n + \delta\vec{x})$$

In the simplest case of a free particle,

$$H = -\frac{1}{2m}\nabla^2 = -\frac{1}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \quad (40)$$

From Equation (40) it is clear that

$$H(\vec{x}'_1, \vec{x}'_2, \dots, \vec{x}'_n) = H(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) \quad (41)$$

which is true for any general closed system.

❖ The Hamiltonian is *invariant* under the *translation operator* \hat{D} , which is defined as an action onto an arbitrary wavefunction $\psi(\vec{x})$ such that

$$\hat{D}\psi(\vec{x}) \equiv \psi(\vec{x} + \delta\vec{x}) \quad (42)$$

For a single-particle state $\psi'(\vec{x}) = H(\vec{x})\psi(\vec{x})$, from (42) one obtains:

$$\hat{D}\psi'(\vec{x}) = \psi'(\vec{x} + \delta\vec{x}) = H(\vec{x} + \delta\vec{x})\psi(\vec{x} + \delta\vec{x})$$

Since the Hamiltonian is invariant under translation,

$\hat{D}\psi'(\vec{x}) = H(\vec{x})\psi(\vec{x} + \delta\vec{x})$, and using the definitions once again,

$$\hat{D}H(\vec{x})\psi(\vec{x}) = H(\vec{x})\hat{D}\psi(\vec{x}) \quad (43)$$

❖ It is said that \hat{D} *commutes* with Hamiltonian

(a standard notation for this is $[\hat{D}, H] \equiv \hat{D}H - H\hat{D} = 0$)

Since $\delta\vec{x}$ is an infinitely small quantity, translation (42) can be expanded:

$$\psi(\vec{x} + \delta\vec{x}) = \psi(\vec{x}) + \delta\vec{x} \cdot \nabla\psi(\vec{x}) \quad (44)$$

Form (44) includes explicitly the momentum operator $\hat{p} = -i\nabla$, hence the translation operator \hat{D} can be rewritten as

$$\hat{D} = 1 + i\delta\vec{x} \cdot \hat{p} \quad (45)$$

Substituting (45) to (43), one obtains

$$[\hat{p}, H] = 0 \quad (46)$$

which is simply the *momentum conservation law* for a single-particle state whose Hamiltonian is invariant under translation.

Generalization of (45) and (46) for the case of multiparticle state leads to the general momentum conservation law for the total momentum

$$\vec{P} = \sum_{i=1}^n \vec{P}_i$$

Rotational invariance

- ❖ When a closed system of particles is rotated about its centre-of-mass, its physical properties remain unchanged

Under a rotation about e.g. z-axis through an angle θ , coordinates x_i, y_i, z_i transform to new coordinates x'_i, y'_i, z'_i as follows:

$$\begin{aligned}x'_i &= x_i \cos \theta - y_i \sin \theta \\y'_i &= x_i \sin \theta + y_i \cos \theta \\z'_i &= z\end{aligned}\tag{47}$$

Correspondingly, the new Hamiltonian of the rotated system will be the same as the initial one, $H(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) = H(\vec{x}'_1, \vec{x}'_2, \dots, \vec{x}'_n)$

Considering rotation through an infinitesimal angle $\delta\theta$, equations (47) transform to

$$x' = x - y\delta\theta, \quad y' = y + x\delta\theta, \quad z' = z$$

A rotational operator \hat{R}_Z is introduced by analogy with the translation operator \hat{D} :

$$\hat{R}_Z \psi(\vec{x}) \equiv \psi(\vec{x}') = \psi(x - y\delta\theta, y + x\delta\theta, z) \quad (48)$$

Expansion to first order in $\delta\theta$ gives:

$$\psi(\vec{x}') = \psi(\vec{x}) - \delta\theta \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \psi(\vec{x}) = (1 + i\delta\theta \hat{L}_Z) \psi(\vec{x})$$

where \hat{L}_Z is z-component of the orbital angular momentum operator \hat{L} :

$$\hat{L}_Z = -i \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \quad (\text{in classical mechanics } \vec{L} = \vec{r} \times \vec{p} \Rightarrow L_z = (xp_y - yp_x))$$

- ☉ For a general case of rotation about an arbitrary direction specified by a unit vector \vec{n} , L_Z has to be replaced by the corresponding projection of L : $\hat{L} \cdot \vec{n}$, giving

$$\hat{R}_n = 1 + i\delta\theta (\hat{L} \cdot \vec{n}) \quad (49)$$

Considering \hat{R}_n acting on a single-particle state $\psi'(\vec{x}) = H(\vec{x})\psi(\vec{x})$ and repeating same steps as for the translation case, one gets:

$$[\hat{R}_n, H] = 0 \quad (50)$$

$$[\hat{L}, H] = 0 \quad (51)$$

This applies to a spin-0 particle moving in a central potential, i.e., in a field that does not depend on a direction, but only on the absolute distance.

❖ If a particle possesses a non-zero spin, the total angular momentum is the sum of the orbital and spin angular momenta:

$$\hat{J} = \hat{L} + \hat{S} \quad (52)$$

and the wavefunction is a product of the independent space wavefunction $\psi(\vec{x})$ and spin wavefunction χ :

$$\Psi = \psi(\vec{x})\chi$$

For the case of spin-1/2 particles, the spin operator is represented in terms of Pauli matrices σ :

$$\hat{S} = \frac{1}{2}\sigma \quad (53)$$

where σ has components (recall Chapter I.):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (54)$$

Let us denote now spin wavefunction for spin “up” state as $\chi = \alpha$ ($S_z = 1/2$) and for spin “down” state as $\chi = \beta$ ($S_z = -1/2$), so that

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (55)$$

Both α and β satisfy the eigenvalue equations for operator (53):

$$\hat{S}_z\alpha = \frac{1}{2}\alpha, \hat{S}_z\beta = -\frac{1}{2}\beta$$

Analogously to (49), rotation operator for a spin-1/2 particle generalizes to

$$\hat{R}_n = 1 + i\delta\theta(\hat{J} \cdot \vec{n}) \quad (56)$$

When the rotation operator \hat{R}_n acts onto a wave function $\Psi = \psi(\vec{x})\chi$, components \hat{L} and \hat{S} of \hat{J} act independently upon the corresponding wave functions:

$$\hat{J}\Psi = (\hat{L} + \hat{S})\psi(\vec{x})\chi = [\hat{L}\psi(\vec{x})]\chi + \psi(\vec{x})[\hat{S}\chi]$$

☉ That means that although the total angular momentum has to be conserved,

$$[\hat{J}, H] = 0$$

the rotational invariance does not in general lead to the conservation of \hat{L} and \hat{S} separately:

$$[\hat{L}, H] = -[\hat{S}, H] \neq 0$$

However, presuming that the forces can change only orientation of the spin, but not its absolute value, one can conclude that

$$[H, \hat{L}^2] = [H, \hat{S}^2] = 0$$

- ❖ Good quantum numbers are those which are associated with conserved observables (operators commute with the Hamiltonian)

Spin is one of the quantum numbers which characterize any particle – elementary or composite.

☉ Spin of a composite particle is the total angular momentum \vec{J} of its constituents in their centre-of-mass frame

- Quarks are spin-1/2 particles \Rightarrow the spin quantum number J of hadrons can be either integer or half-integer
- Spin projections on a chosen z-axis – J_z – can take any of $2J+1$ values, from $-J$ to J with the “step” of 1, depending on the particle’s spin orientation
- ☉ Usually, it is assumed that L and S are “good” quantum numbers together with J , while J_z depends on the spin orientation.

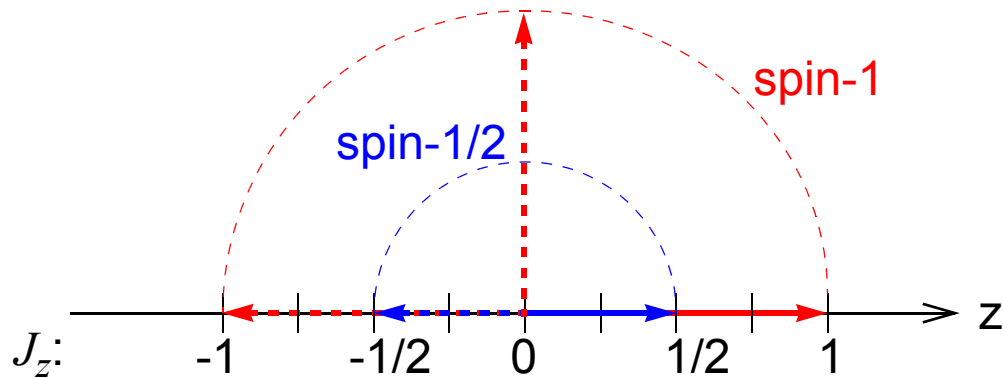


Figure 70: A naive illustration of possible J_z values for spin-1/2 and spin-1 particles

Using “good” quantum numbers, one can refer to a particle via *spectroscopic notation*, like

$${}^{2S+1}L_J \quad (57)$$

- ⊙ Following chemistry traditions, instead of numerical values of $L=0,1,2,3\dots$, letters S,P,D,F... are used correspondingly
- ⊙ In this notation, the lowest-lying ($L=0$) bound state of two particles of spin-1/2 (a meson) will be 1S_0 or 3S_1

$$\begin{array}{cc}
 L=0 & \begin{array}{c} {}^1S_0 \\ \uparrow \quad \downarrow \\ S=1/2-1/2=0 \\ J=L+S=0 \end{array} & \begin{array}{c} {}^3S_1 \\ \uparrow \quad \uparrow \\ S=1/2+1/2=1 \\ J=L+S=1 \end{array}
 \end{array}$$

Figure 71: Quark-antiquark states for $L=0$

☉ For mesons with $L \geq 1$, possible states are: ${}^1L_L, {}^3L_{L+1}, {}^3L_L, {}^3L_{L-1}$

❖ Baryons are bound states of 3 quarks \Rightarrow there are two orbital angular momenta connected to the relative motion of quarks.

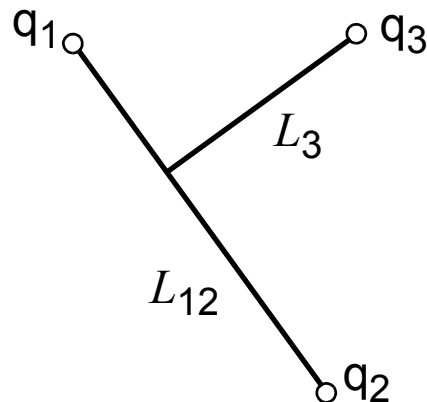


Figure 72: Internal orbital angular momenta of a three-quark state

⊙ total orbital angular momentum is $L=L_{12}+L_3$.

⊙ spin of a baryon $S=S_1+S_2+S_3 \Rightarrow S=1/2$ or $S=3/2$

Possible baryon states:

$${}^2S_{1/2}, {}^4S_{3/2} \quad (L = 0)$$

$${}^2P_{1/2}, {}^2P_{3/2}, {}^4P_{1/2}, {}^4P_{3/2}, {}^4P_{5/2} \quad (L = 1)$$

$${}^2L_{L+1/2}, {}^2L_{L-1/2}, {}^4L_{L-3/2}, {}^4L_{L-1/2}, {}^4L_{L+1/2}, {}^4L_{L+3/2} \quad (L \geq 2)$$

Parity

❖ Parity transformation is the transformation by reflection:

$$\vec{x}_i \rightarrow \vec{x}'_i = -\vec{x}_i \quad (58)$$

A system is said to be invariant under parity transformation if

$$H(-\vec{x}_1, -\vec{x}_2, \dots, -\vec{x}_n) = H(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n)$$

❖ Parity is not an exact symmetry: it is violated in weak interactions!

☉ Absolute handedness can actually be defined

A parity operator \hat{P} is defined as

$$\hat{P}\psi(\vec{x}, t) \equiv P_a \psi(-\vec{x}, t) \quad (59)$$

Two consecutive reflections must result in a system identical to the initial:

$$\hat{P}^2 \psi(\vec{x}, t) = \psi(\vec{x}, t) \quad (60)$$

☉ From equations (59) and (60), $P_a = +1, -1$

Consider a particle wavefunction which is a solution of the Dirac equation

(16): $\psi_{\vec{p}}(\vec{x}, t) = u(\vec{p})e^{i(\vec{p}\vec{x} - Et)}$, where $u(\vec{p})$ is a four-component spinor independent of \vec{x} . Parity operation on such a wavefunction is then:

$$\hat{P}\psi_{\vec{p}}(\vec{x}, t) = P_a u(-\vec{p})e^{i((-\vec{p})(-\vec{x}) - Et)} \tag{61}$$

❖ Particle at rest ($\vec{p} = 0$) is an eigenstate of the parity operator:

$$\hat{P}\psi_0(\vec{x}, t) = P_a u(0)e^{-iEt} = P_a \psi_0(\vec{x}, t) \tag{62}$$

☉ Eigenvalue P_a is called the *intrinsic parity* of a particle a : intrinsic parity is parity of a particle at rest

❖ Different particles have different, independent, values of parity P_a . For a system of n particles,

$$\hat{P}\psi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n, t) \equiv P_1 P_2 \dots P_n \psi(-\vec{x}_1, -\vec{x}_2, \dots, -\vec{x}_n, t)$$

Polar coordinates offer a convenient frame: parity transformation is

$$r \rightarrow r' = r, \theta \rightarrow \theta' = \pi - \theta, \varphi \rightarrow \varphi' = \pi + \varphi$$

and a wavefunction can be written as

$$\psi_{nlm}(\vec{x}) = R_{nl}(r) Y_l^m(\theta, \varphi) \quad (63)$$

In Equation (63), R_{nl} is a function of the radius only, and Y_l^m are *spherical harmonics*, which describe angular dependence.

Under the parity transformation, R_{nl} does not change, while spherical harmonics change as

$$Y_l^m(\theta, \varphi) \rightarrow Y_l^m(\pi - \theta, \pi + \varphi) = (-1)^l Y_l^m(\theta, \varphi)$$

⇓

$$\hat{P}\psi_{nlm}(\vec{x}) = P_a \psi_{nlm}(-\vec{x}) = P_a (-1)^l \psi_{nlm}(\vec{x})$$

- ☉ A particle with a definite orbital angular momentum is also an eigenstate of parity with an eigenvalue $P_a (-1)^l$.

Considering only electromagnetic and strong interactions, and using the usual argumentation, one can prove that parity is conserved:

$$[\hat{P}, H] = 0$$

☉ Recall: the Dirac equation (16) suggests a four-component wavefunction to describe both electrons and positrons: 2 components for electrons, 2 components for positrons.

❖ Indeed, intrinsic parities of e^- and e^+ are related, namely:

$$P_{e^+} P_{e^-} = -1$$

This is true for all the fermions (spin-1/2 particles), i.e.,

$$P_f P_{\bar{f}} = -1 \tag{64}$$

Experimentally this can be confirmed by studying the reaction $e^+e^- \rightarrow \gamma\gamma$ where initial state has zero orbital momentum and parity of $P_{e^-} P_{e^+}$

☉ If the final state has relative orbital angular momentum l_γ , its parity is $P_\gamma^2 (-1)^{l_\gamma}$

⊙ Since $P_\gamma^2 = 1$, from the parity conservation law stems that $P_{e^-} P_{e^+} = (-1)^{l_\gamma}$

Experimental measurements of l_γ confirm (64)

While (64) can be proven in experiments, it is impossible to determine P_{e^-} or P_{e^+} , since these particles are created or destroyed only in pairs.

❖ Conventionally defined parities of leptons are:

$$P_{e^-} = P_{\mu^-} = P_{\tau^-} \equiv 1 \quad (65)$$

And consequently, parities of antileptons have opposite sign.

⊙ Since quarks and antiquarks are also produced only in pairs, their parities are defined also by convention:

$$P_u = P_d = P_s = P_c = P_b = P_t = 1 \quad (66)$$

with parities of antiquarks being -1.

For a meson $M=(a\bar{b})$, parity is then calculated as

$$P_M = P_a P_{\bar{b}} (-1)^L = (-1)^{L+1} \quad (67)$$

- ⊙ For the low-lying mesons ($L=0$) this implies parity of -1, which is confirmed by observations

For a baryon $B=(abc)$, parity is given as

$$P_B = P_a P_b P_c (-1)^{L_{12}} (-1)^{L_3} = (-1)^{L_{12} + L_3} \quad (68)$$

and for antibaryon $P_{\bar{B}} = -P_B$, similarly to the case of leptons.

- ⊙ For the low-lying baryons with $L_{12}=L_3=0$, (68) predicts positive parities, which is also confirmed by experiment.

Parity of the photon can be deduced from the classical field theory, considering Poisson's equation:

$$\nabla \cdot \vec{E}(\vec{x}, t) = \frac{1}{\epsilon_0} \rho(\vec{x}, t)$$

Under a parity transformation, charge density changes as $\rho(\vec{x}, t) \rightarrow \rho(-\vec{x}, t)$ and ∇ changes its sign, so that to keep the equation invariant, the electric field must transform as

$$\vec{E}(\vec{x}, t) \rightarrow -\vec{E}(-\vec{x}, t) \quad (69)$$

The electromagnetic field is described by the vector and scalar potentials:

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} \quad (70)$$

For photons, only the vector part corresponds to the wavefunction:

$$\vec{A}(\vec{x}, t) = N\epsilon(\vec{k})e^{i(\vec{k}\vec{x} - Et)}$$

Under parity transformation: $\vec{A}(\vec{x}, t) \rightarrow P_\gamma\vec{A}(-\vec{x}, t)$, and from (69) follows

$$\vec{E}(\vec{x}, t) \rightarrow P_\gamma\vec{E}(-\vec{x}, t). \quad (71)$$

☉ Comparing (71) and (69), one concludes that parity of photon is $P_\gamma = -1$

Charge conjugation

- ❖ Charge conjugation replaces particles by their antiparticles, reversing charges and magnetic moments
- ❖ Charge conjugation is violated in weak interactions
 - ⊙ Absolute sign of the electric charge can actually be defined

For strong and electromagnetic interactions, charge conjugation is a symmetry:

$$[\hat{C}, H] = 0$$

- ⊙ It is convenient now to denote a state in a compact notation, using Dirac's "ket" representation: $|\pi^+, \vec{p}\rangle$ denotes a pion having momentum \vec{p} , or, in general case,

$$|\pi^+ \Psi_1; \pi^- \Psi_2\rangle \equiv |\pi^+ \Psi_1\rangle |\pi^- \Psi_2\rangle \quad (72)$$

- ⊙ Next, we denote particles which have distinct antiparticles with "a", and otherwise – with "α"

In such notations, we describe the action of the charge conjugation operator upon particles of kind “ α ” as:

$$\hat{C}|\alpha, \Psi\rangle = C_\alpha|\alpha, \Psi\rangle \quad (73)$$

meaning that the final state acquires a phase factor C_α , and for “ a ” as:

$$\hat{C}|a, \Psi\rangle = |\bar{a}, \Psi\rangle \quad (74)$$

meaning that from a particle in the initial state we came to the antiparticle in the final state.

Since the consecutive transformation turns antiparticles back to particles,

$\hat{C}^2 = 1$ and hence

$$C_\alpha = \pm 1 \quad (75)$$

For multiparticle states the transformation is:

$$\hat{C}|\alpha_1, \alpha_2, \dots, a_1, a_2, \dots; \Psi\rangle = C_{\alpha_1} C_{\alpha_2} \dots |\alpha_1, \alpha_2, \dots, \bar{a}_1, \bar{a}_2, \dots; \Psi\rangle \quad (76)$$

❖ From (73) follows that particles $\alpha=\gamma,\pi^0,\dots$ are eigenstates of \hat{C} with eigenvalues $C_\alpha=\pm 1$.

❖ Other eigenstates can be constructed from particle-antiparticle pairs:

$$\hat{C}|a, \Psi_1; \bar{a}, \Psi_2\rangle = |\bar{a}, \Psi_1; a, \Psi_2\rangle = \pm |a, \Psi_1; \bar{a}, \Psi_2\rangle$$

⊙ For a state of definite orbital angular momentum, interchanging between particle and antiparticle reverses their relative position vector, for example:

$$\hat{C}|\pi^+ \pi^-; L\rangle = (-1)^L |\pi^+ \pi^-; L\rangle \quad (77)$$

⊙ For fermion-antifermion pairs theory predicts

$$\hat{C}|f\bar{f}; J, L, S\rangle = (-1)^{L+S} |f\bar{f}; J, L, S\rangle \quad (78)$$

This implies that e.g. a neutral pion π^0 , being a 1S_0 state of $u\bar{u}$ and $d\bar{d}$, must have C-parity of 1.

Tests of C-invariance

- ❖ Prediction of $C_{\pi^0} = 1$ can be confirmed experimentally by observing the decay $\pi^0 \rightarrow \gamma\gamma$.

The final state has $C=1$, and from the relations

$$\hat{C}|\pi^0\rangle = C_{\pi^0}|\pi^0\rangle$$

$$\hat{C}|\gamma\gamma\rangle = C_\gamma C_\gamma |\gamma\gamma\rangle = |\gamma\gamma\rangle$$

follows that $C_{\pi^0} = 1$.

- ❖ C_γ can be inferred from the classical field theory:

$$\vec{A}(\vec{x}, t) \rightarrow C_\gamma \vec{A}(\vec{x}, t)$$

under the charge conjugation, and since all electric charges swap, electric field and scalar potential also change sign:

$$\vec{E}(\vec{x}, t) \rightarrow -\vec{E}(\vec{x}, t), \quad \phi(\vec{x}, t) \rightarrow -\phi(\vec{x}, t)$$

Upon substitution into (70) this gives $C_\gamma = -1$.

❖ To check predictions of the C-invariance and of the value of C_γ , one can try to look for the decay

$$\pi^0 \rightarrow \gamma + \gamma + \gamma$$

☉ If predictions for C_γ and C_{π^0} are true, this mode should be **forbidden**:

$$\hat{C}|\gamma\gamma\gamma\rangle = (C_\gamma)^3|\gamma\gamma\gamma\rangle = -|\gamma\gamma\gamma\rangle$$

contradicts all previous observations. Indeed, experimentally, this 3γ mode has never been observed.

❖ Symmetry requirements and corresponding conservation laws explain why certain particle decays are never observed – forbidden

Another confirmation of C-invariance comes from observation of η -meson decays:

$$\eta \rightarrow \gamma + \gamma$$

$$\eta \rightarrow \pi^0 + \pi^0 + \pi^0$$

$$\eta \rightarrow \pi^+ + \pi^- + \pi^0$$

- ◎ They are electromagnetic decays, and first two clearly indicate that $C_\eta=1$. Identical charged pions momenta distribution in the last confirms C-invariance.