IV. Space-time symmetries

- Conservation laws have their origin in symmetries and invariance properties of the underlying interactions
  - Exact symmetry implies a conservation law $\Rightarrow$ an observable which *absolute* value can not be defined ("*non-observable*")

Symmetries, conservation laws and "non-observables":

Symmetry transformation	Conservation law or selection rule	Non-observable
Space translation: $x \to x+\delta x$	momentum	absolute spatial position
Rotation: $\bar{x} \to \bar{x}'$	angular momentum	absolute spatial direction
Time translation: $t \to t+\delta t$	energy	absolute time
Reflection: $\bar{x} \to -\bar{x}$	parity	"handedness" (absolute generalized right/left)
Charge conjugation: $q \to -q$	particle-antiparticle symmetry	absolute sign of electric charge
$\psi \to e^{iq\theta}\psi$	charge q	relative phase between states of different q
$\psi \to e^{iL\theta}\psi$	lepton number L	relative phase between states of different L
$\psi \to e^{iB\theta}\psi$	baryon number B	relative phase between states of different B

Translational invariance

- When a closed system of particles is moved from from one position in space to another, its physical properties do not change

Considering an infinitesimal translation $\vec{x}_i \rightarrow \vec{x}'_i = \vec{x}_i + \delta\vec{x}$, the Hamiltonian of the system transforms as:

$$H(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n) \rightarrow H(\vec{x}_1 + \delta\vec{x}, \vec{x}_2 + \delta\vec{x}, \ldots, \vec{x}_n + \delta\vec{x})$$

In the simplest case of a free particle,

$$H = -\frac{1}{2m}\nabla^2 = -\frac{1}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \qquad (39)$$

From Equation (39) it is clear that

$$H(\vec{x}'_1, \vec{x}'_2, \ldots, \vec{x}'_n) = H(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n) \qquad (40)$$

which is true for any general closed system.

- The Hamiltonian is *invariant* under the *translation operator* $\hat{D}$, which is defined as an action onto an arbitrary wavefunction $\psi(\vec{x})$ such that

$$\hat{D}\psi(\vec{x}) \equiv \psi(\vec{x} + \delta\vec{x}) \tag{41}$$

For a single-particle state $\psi'(\vec{x}) = H(\vec{x})\psi(\vec{x})$, from (41) one obtains:

$$\hat{D}\psi'(\vec{x}) = \psi'(\vec{x} + \delta\vec{x}) = H(\vec{x} + \delta\vec{x})\psi(\vec{x} + \delta\vec{x})$$

Since the Hamiltonian is invariant under translation, $\hat{D}\psi'(\vec{x}) = H(\vec{x})\psi(\vec{x} + \delta\vec{x})$, and using the definitions once again,

$$\hat{D}H(\vec{x})\psi(\vec{x}) = H(\vec{x})\hat{D}\psi(\vec{x}) \tag{42}$$

- It is said that $\hat{D}$ *commutes* with Hamiltonian (a standard notation for this is $[\hat{D}, H] \equiv \hat{D}H - H\hat{D} = 0$)

Since $\delta\vec{x}$ is an infinitely small quantity, translation (41) can be expanded:

$$\psi(\vec{x} + \delta\vec{x}) = \psi(\vec{x}) + \delta\vec{x} \cdot \nabla\psi(\vec{x}) \tag{43}$$

Form (43) includes explicitly the momentum operator $\hat{p} = -i\nabla$, hence the translation operator $\hat{D}$ can be rewritten as

$$\hat{D} = 1 + i\delta\vec{x} \cdot \hat{p} \tag{44}$$

Substituting (44) to (42), one obtains

$$[\hat{p}, H] = 0 \tag{45}$$

which is simply the *momentum conservation law* for a single-particle state whose Hamiltonian in invariant under translation.

Generalization of (44) and (45) for the case of multiparticle state leads to the general momentum conservation law for the total momentum

$$\vec{p} = \sum_{i = 1}^{n} \vec{p}_i$$

Rotational invariance

- When a closed system of particles is rotated about its centre-of-mass, its physical properties remain unchanged

Under a rotation about e.g. z-axis through an angle θ, coordinates x_i, y_i, z_i transform to new coordinates x'_i, y'_i, z'_i as follows:

$$\begin{aligned} x'_i &= x_i \cos\theta - y_i \sin\theta \\ y'_i &= x_i \sin\theta + y_i \cos\theta \\ z'_i &= z \end{aligned} \tag{46}$$

Correspondingly, the new Hamiltonian of the rotated system will be the same as the initial one, $H(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n) = H(\vec{x}'_1, \vec{x}'_2, \ldots, \vec{x}'_n)$

Considering rotation through an infinitesimal angle $\delta\theta$, equations (46) transform to

$$x' = x - y\delta\theta\ , y' = y + x\delta\theta\ , z' = z$$

A rotational operator $\hat{R}_Z$ is introduced by analogy with the translation operator $\hat{D}$:

$$\hat{R}_z\psi(\vec{x}) \equiv \psi(\vec{x}') = \psi(x - y\delta\theta, y + x\delta\theta, z) \tag{47}$$

Expansion to first order in $\delta\theta$ gives

$$\psi(\vec{x}') = \psi(\vec{x}) - \delta\theta\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\psi(\vec{x}) = (1 + i\delta\theta\hat{L}_z)\psi(\vec{x})$$

where $\hat{L}_z$ is z-component of the orbital angular momentum operator $\hat{L}$:

$\hat{L}_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$ (in classical mechanics $\vec{L} = \vec{r} \times \vec{p} \Rightarrow L_z = (xp_y - yp_x)$)

- For a general case of rotation about an arbitrary direction specified by a unit vector $\vec{n}$, $\hat{L}_Z$ has to be replaced by the corresponding projection of $\hat{L}$: $\hat{L} \cdot \vec{n}$, giving

$$\hat{R}_n = 1 + i\delta\theta(\hat{L} \cdot \vec{n}) \tag{48}$$

Considering $\hat{R}_n$ acting on a single-particle state $\psi'(\vec{x}) = H(\vec{x})\psi(\vec{x})$ and repeating same steps as for the translation case, one gets:

$$[\hat{R}_n, H] = 0 \qquad (49)$$

$$[\hat{L}, H] = 0 \qquad (50)$$

This applies to a spin-0 particle moving in a central potential, i.e., in a field that does not depend on a direction, but only on the absolute distance.

- If a particle posseses a non-zero spin, the total angular momentum is the sum of the orbital and spin angular momenta:

$$\hat{J} = \hat{L} + \hat{S} \qquad (51)$$

  and the wavefunction is a product of the independent space wavefuncion $\psi(\vec{x})$ and spin wavefunction χ:

$$\Psi = \psi(\vec{x})\chi$$

For the case of spin-1/2 particles, the spin operator is represented in terms of Pauli matrices σ:

$$\hat{S} = \frac{1}{2}\sigma \tag{52}$$

where σ has components (recall Chapter I.):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{53}$$

Let us denote now spin wavefunction for spin "up" state as $\chi = \alpha$ ($S_z = 1/2$) and for spin "down" state as $\chi = \beta$ ($S_z = -1/2$), so that

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{54}$$

Both α and β satisfy the eigenvalue equations for operator (52):

$$\hat{S}_z\alpha = \frac{1}{2}\alpha , \hat{S}_z\beta = -\frac{1}{2}\beta$$

Analogously to (48), rotation operator for a spin-1/2 particle generalizes to

$$\hat{R}_n = 1 + i\delta\theta(\hat{J} \cdot \vec{n}) \quad (55)$$

When the rotation operator $\hat{R}_n$ acts onto a wave function $\Psi = \psi(\vec{x})\chi$, components $\hat{L}$ and $\hat{S}$ of $\hat{J}$ act independently upon the corresponding wave functions:

$$\hat{J}\Psi = (\hat{L} + \hat{S})\psi(\vec{x})\chi = [\hat{L}\psi(\vec{x})]\chi + \psi(\vec{x})[\hat{S}\chi]$$

- That means that although the total angular momentum has to be conserved,

  $$[\hat{J}, H] = 0$$

  the rotational invariance does not in general lead to the conservation of $\hat{L}$ and $\hat{S}$ separately:

  $$[\hat{L}, H] = -[\hat{S}, H] \neq 0$$

However, presuming that the forces can change only orientation of the spin, but not its absolute value, one can conclude that

$$[H, \hat{L}^2] = [H, \hat{S}^2] = 0$$

- Good quantum numbers are those which are associated with conserved observables (operators commute with the Hamiltonian)

Spin is one of the quantum numbers which characterize any particle – elementary or composite.

- Spin $\vec{S}_P$ of a composite particle is the total angular momentum $\vec{J}$ of its constituents in their centre-of-mass frame

– Quarks are spin-1/2 particles $\Rightarrow$ the spin quantum number $S_P = J$ of hadrons can be either integer or half-integer

– Spin projections on a chosen z-axis – J_z – can take any of $2J+1$ values, from $-J$ to J with the “step” of 1, depending on the particle’s spin orientation

- Usually, it is assumed that L and S are “good” quantum numbers together with $J = S_P$, while J_z depends on the spin orientation.

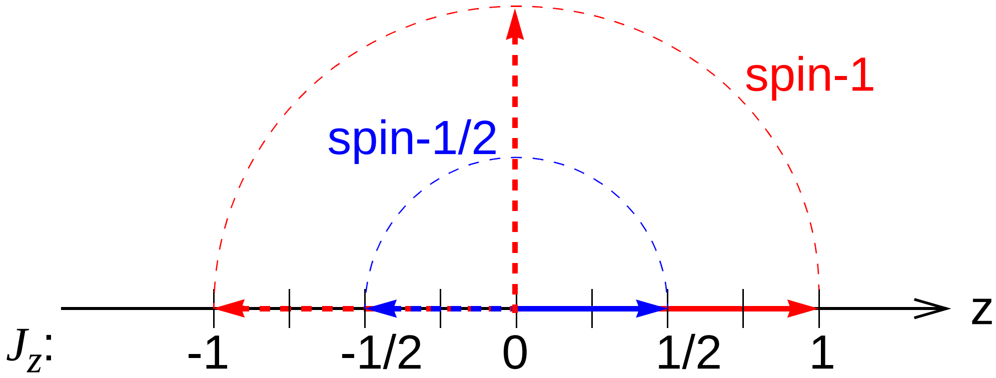

Figure 40: A naive illustration of possible J_z values for spin-1/2 and spin-1 particles

Using "good" quantum numbers, one can refer to a particle via *spectroscopic notation*, like

$$^{2S+1}L_J \tag{56}$$

- Following chemistry traditions, instead of numerical values of $L=0,1,2,3...$, letters S,P,D,F... are used correspondingly
- In this notation, the lowest-lying ($L=0$) bound state of two particles of spin-1/2 (a meson) will be $^1S_0$ or $^3S_1$

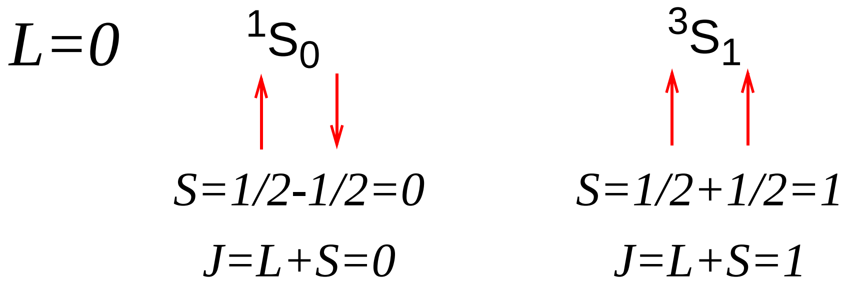

Figure 41: Quark-antiquark states for $L=0$

- For mesons with $L \geq 1$, possible states are: ^1L_L, $^3L_{L+1}$, ^3L_L, $^3L_{L-1}$

- Baryons are bound states of 3 quarks $\Rightarrow$ there are two orbital angular momenta connected to the relative motion of quarks.

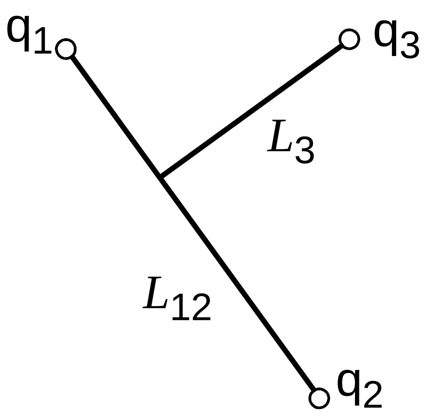

Figure 42: Internal orbital angular momenta of a three-quark state

- total orbital angular momentum is $L=L_{12}+L_3$.
- spin of a baryon $S=S_1+S_2+S_3 \Rightarrow S=1/2$ or $S=3/2$

Possible baryon states:

$^2S_{1/2}$, $^4S_{3/2}$ $(L = 0)$

$^2P_{1/2}$, $^2P_{3/2}$, $^4P_{1/2}$, $^4P_{3/2}$, $^4P_{5/2}$ $(L = 1)$

$^2L_{L+1/2}$, $^2L_{L-1/2}$, $^4L_{L-3/2}$, $^4L_{L-1/2}$, $^4L_{L+1/2}$, $^4L_{L+3/2}$ $(L \geq 2)$

Parity

- Parity transformation is the transformation by reflection:

$$\vec{x}_i \to \vec{x}'_i = -\vec{x}_i \tag{57}$$

A system is said to be invariant under parity transformation if $H(-\vec{x}_1, -\vec{x}_2, \ldots, -\vec{x}_n) = H(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n)$

- Parity is not an exact symmetry: it is violated in weak interactions!
  - Absolute handedness can actually be defined

A parity operator $\hat{P}$ is defined as

$$\hat{P}\psi(\vec{x}, t) \equiv P_a \psi(-\vec{x}, t) \tag{58}$$

Two consecutive reflections must result in the identical to initial system:

$$\hat{P}^2\psi(\vec{x}, t) = \psi(\vec{x}, t) \tag{59}$$

  - From equations (58) and (59), $P_a = +1\ ,\ -1$

Consider a particle wavefunction which is a solution of the Dirac equation (16): $\psi_{\vec{p}}(\vec{x}, t) = u(\vec{p})e^{i(\vec{p}\vec{x} - Et)}$, where $u(\bar{p})$ is a four-component spinor independent of $\bar{x}$. Parity operation on such a wavefunction is then:

$$\hat{P}\psi_{\vec{p}}(\vec{x}, t) = P_a u(-\vec{p})e^{i((-\vec{p})(-\vec{x}) - Et)} \quad (60)$$

❖ Particle at rest ($\vec{p} = 0$) is an eigenstate of the parity operator:

$$\hat{P}\psi_0(\vec{x}, t) = P_a u(0)e^{-iEt} = P_a\psi_0(\vec{x}, t) \quad (61)$$

- Eigenvalue P_a is called the *intrinsic parity* of a particle a: intrinsic parity is parity of a particle at rest

❖ Different particles have different, independent, values of parity P_a. For a system of n particles,

$$\hat{P}\psi(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n, t) \equiv P_1 P_2 \ldots P_n \psi(-\vec{x}_1, -\vec{x}_2, \ldots, -\vec{x}_n, t)$$

Polar coordinates offer a convenient frame: parity transformation is

$$r \to r' = r\,,\ \theta \to \theta' = \pi - \theta\,,\ \varphi \to \varphi' = \pi + \varphi$$

and a wavefunction can be written as

$$\psi_{nlm}(\vec{x}) = R_{nl}(r) Y_l^m(\theta, \varphi) \tag{62}$$

In Equation (62), R_{nl} is a function of the radius only, and Y_l^m are *spherical harmonics*, which describe angular dependence.

Under the parity transformation, R_{nl} does not change, while spherical harmonics change as

$$Y_l^m(\theta, \varphi) \to Y_l^m(\pi - \theta, \pi + \varphi) = (-1)^l Y_l^m(\theta, \varphi)$$

$$\Downarrow$$

$$\hat{P}\psi_{nlm}(\vec{x}) = P_a \psi_{nlm}(-\vec{x}) = P_a(-1)^l \psi_{nlm}(\vec{x})$$

- A particle with a definite orbital angular momentum is also an eigenstate of parity with an eigenvalue $P_a(-1)^l$.

Considering only electromagnetic and strong interactions, and using the usual argumentation, one can prove that parity is conserved:

$$[\hat{P}, H] = 0$$

- Recall: the Dirac equation (16) suggests a four-component wavefunction to describe both electrons and positrons: 2 components for electrons, 2 components for positrons.

- Indeed, intrinsic parities of e^- and e^+ are related, namely:

$$P_{e^+} P_{e^-} = -1$$

This is true for all the fermions (spin-1/2 particles), i.e.,

$$P_f P_{\bar{f}} = -1 \qquad (63)$$

Experimentally this can be confirmed by studying the reaction $e^+e^- \to \gamma\gamma$ where initial state has zero orbital momentum and parity of $P_{e^-} P_{e^+}$.

- If the final state has relative orbital angular momentum l_γ, its parity is $P_\gamma^2 (-1)^{l_\gamma}$.

Since $P_\gamma^2 = 1$, from the parity conservation law stems that $P_{e^-} P_{e^+} = (-1)^{l_\gamma}$

Experimental measurements of l_γ confirm (63)

While (63) can be proved in experiments, it is impossible to determine P_{e^-} or P_{e^+}, since these particles are created or destroyed only in pairs.

- Conventionally defined parities of leptons are:

$$P_{e^-} = P_{\mu^-} = P_{\tau^-} \equiv 1 \qquad (64)$$

And consequently, parities of antileptons have opposite sign.

Since quarks and antiquarks are also produced only in pairs, their parities are defined also by convention:

$$P_u = P_d = P_s = P_c = P_b = P_t = 1 \qquad (65)$$

with parities of antiquarks being -1.

For a meson M=(a$\bar{b}$), parity is then calculated as

$$P_M = P_a P_{\bar{b}} (-1)^L = (-1)^{L+1} \qquad (66)$$

- For the low-lying mesons ($L=0$) this implies parity of -1, which is confirmed by observations

For a baryon B=(abc), parity is given as

$$P_B = P_a P_b P_c (-1)^{L_{12}} (-1)^{L_3} = (-1)^{L_{12}+L_3} \qquad (67)$$

and for antibaryon $P_{\bar{B}} = -P_B$, similarly to the case of leptons.

- For the low-lying baryons with $L_{12}=L_3=0$, (67) predicts positive parities, which is also confirmed by experiment.

Parity of the photon can be deduced from the classical field theory, considering Poisson's equation:

$$\nabla \cdot \vec{E}(\vec{x}, t) = \frac{1}{\varepsilon_0} \rho(\vec{x}, t)$$

Under a parity transformation, charge density changes as $\rho(\vec{x}, t) \to \rho(-\vec{x}, t)$ and ∇ changes its sign, so that to keep the equation invariant, the electric field must transform as

$$\vec{E}(\vec{x}, t) \to -\vec{E}(-\vec{x}, t) \tag{68}$$

The electromagnetic field is described by the vector and scalar potentials:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \tag{69}$$

For photons, only the vector part corresponds to the wavefunction:

$$\vec{A}(\vec{x}, t) = N\vec{\varepsilon}(\vec{k})e^{i(\vec{k}\vec{x} - Et)}$$

Under parity transformation: $\vec{A}(\vec{x}, t) \to P_\gamma \vec{A}(-\vec{x}, t)$, and from (68) follows

$$\vec{E}(\vec{x}, t) \to P_\gamma \vec{E}(-\vec{x}, t). \tag{70}$$

- Comparing (70) and (68), one concludes that parity of photon is $P_\gamma = -1$

Charge conjugation

- Charge conjugation replaces particles by their antiparticles, reversing charges and magnetic moments

- Charge conjugation is violated in weak interactions
  - Absolute sign of the electric charge can actually be defined

For strong and electromagnetic interactions, charge conjugation is a symmetry:

$$[\hat{C}, H] = 0$$

- It is convenient now to denote a state in a compact notation, using Dirac's "ket" representation: $|\pi^+, \vec{p}\rangle$ denotes a pion having momentum $\vec{p}$, or, in general case,

$$|\pi^+\Psi_1;\pi^-\Psi_2\rangle \equiv |\pi^+\Psi_1\rangle|\pi^-\Psi_2\rangle \tag{71}$$

- Next, we denote particles which have distinct antiparticles with "a" , and otherwise – with "α"

In such notations, we describe the action of the charge conjugation operator upon particles of kind "α" as:

$$\hat{C}|\alpha, \Psi\rangle = C_\alpha|\alpha, \Psi\rangle \quad (72)$$

meaning that the final state acquires a phase factor C_α, and for "a" as:

$$\hat{C}|a, \Psi\rangle = |\bar{a}, \Psi\rangle \quad (73)$$

meaning that from a particle in the initial state we came to the antiparticle in the final state.

Since the consequtive transformation turns antiparticles back to particles, $\hat{C}^2 = 1$ and hence

$$C_\alpha = \pm 1 \quad (74)$$

For multiparticle states the transformation is:

$$\hat{C}|\alpha_1, \alpha_2, \ldots, a_1, a_2, \ldots;\Psi\rangle = C_{\alpha_1}C_{\alpha_2}\ldots|\alpha_1, \alpha_2, \ldots, \bar{a}_1, \bar{a}_2, \ldots;\Psi\rangle \quad (75)$$

- From (72) follows that particles $\alpha=\gamma,\pi^0,\ldots$ are eigenstates of $\hat{C}$ with eigenvalues $C_\alpha=\pm1$.

- Other eigenstates can be constructed from particle-antiparticle pairs:

$$\hat{C}|a, \Psi_1;\bar{a}, \Psi_2\rangle = |\bar{a}, \Psi_1;a\Psi_2\rangle = \pm|a, \Psi_1;\bar{a}, \Psi_2\rangle$$

  - For a state of definite orbital angular momentum, interchanging between particle and antiparticle reverses their relative position vector, for example:

$$\hat{C}|\pi^+\pi^-;L\rangle = (-1)^L|\pi^+\pi^-;L\rangle \tag{76}$$

  - For fermion-antifermion pairs theory predicts

$$\hat{C}|f\bar{f};J, L, S\rangle = (-1)^{L+S}|f\bar{f};J, L, S\rangle \tag{77}$$

This implies that e.g. a neutral pion π^0, being a $^1S_0$ state of $u\bar{u}$ and $d\bar{d}$, must have C-parity of 1.

Tests of C-invariance

- Prediction of $C_{\pi^0} = 1$ can be confirmed experimentally by observing the decay $\pi^0 \rightarrow \gamma\gamma$.

The final state has $C=1$, and from the relations

$$\hat{C}|\pi^0\rangle = C_{\pi^0}|\pi^0\rangle$$

$$\hat{C}|\gamma\gamma\rangle = C_\gamma C_\gamma|\gamma\gamma\rangle = |\gamma\gamma\rangle$$

follows that $C_{\pi^0} = 1$.

- C_γ can be inferred from the classical field theory:

$$\vec{A}(\vec{x}, t) \rightarrow C_\gamma \vec{A}(\vec{x}, t)$$

under the charge conjugation, and since all electric charges swap, electric field and scalar potential also change sign:

$$\vec{E}(\vec{x}, t) \to -\vec{E}(\vec{x}, t)\ ,\ \phi(\vec{x}, t) \to -\phi(\vec{x}, t)$$

Upon substitution into (69) this gives $C_\gamma = -1$.

- To check predictions of the C-invariance and of the value of C_γ, one can try to look for the decay

$$\pi^0 \to \gamma + \gamma + \gamma$$

  - If predictions for C_γ and C_{π^0} are true, this mode should be forbidden:

$$\hat{C}|\gamma\gamma\gamma\rangle = (C_\gamma)^3|\gamma\gamma\gamma\rangle = -|\gamma\gamma\gamma\rangle$$

contradicts all previous observations. Indeed, experimentally, this 3γ mode has never been observed.

- Symmetry requirements and corresponding conservation laws explain why certain particle decays are never observed – forbidden

Another confirmation of C-invariance comes from observation of η-meson decays:

$$\eta \rightarrow \gamma + \gamma$$
$$\eta \rightarrow \pi^{0} + \pi^{0} + \pi^{0}$$
$$\eta \rightarrow \pi^{+} + \pi^{-} + \pi^{0}$$

- They are electromagnetic decays, and first two clearly indicate that C_η=1. Identical charged pions momenta distribution in the last confirms C-invariance.