Relativistic Quantum Mechanics and Quantum Field Theory (QFT)

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Abstract

Introductory chapter and the five PBL cycles for the coures QFT 7.5 ECTS credits

Introductory Chapter

1 Historical Introduction

Quantum mechanics was put on a sound footing with work of Schrödinger and Heisenberg in 1925 and the basic formalism was worked out by them and Dirac, Born, Jordan,... in the following years. This solved the problem of how particles move in external potentials and forms the basic of all atomic physics and chemistry.

Soon after Dirac and Heisenberg attacked the question of the quantization of the radiation field. The basic formalism followed from using the Hamiltonian for the electromagnetic field and the mathematics that had been developed for quantum mechanics. At the same time Dirac made a relativistic version of the Schrödinger equation. This equation solved one problem, that of finding a positive definite current but in turn introduced other problems. In particular, there was no lower bound to the energies. This Dirac finally reinterpreted by introducing what we now know as the "Dirac sea" and a major consequence was the prediction of anti-particles.

However, the combined theory of the quantized Dirac equation and the radiation field turned out to be harder than expected. Many calculations were performed in the 1930s but when the perturbation theory was pushed to higher orders the equations stopped to make sense. The calculations were plagued with infinities, also called divergences. The solution to this problem was done independently by Feynman, Schwinger and Tomonaga, afeat for which they received the 1965 Nobel prize. The lectures given by them at that occasion still make interesting reading and are accessible via `http://nobelprize.org`. I would however wait with reading them till after you have finished the course. It is better to learn about all the wrong turns one can take after you have learned the correct road.

This course will finish essentially with their results. Field theory itself is a much broader subject as you can easily judge from the size of the book which is the main literature source for this course. We will not treat all the developments which have happened afterwards nor will we show alternative quantization methods beyond the canonical quantization.

2 Course Overview

2.1 The PBL cycles

This course consists of five Problem Based Learning (PBL) cycles. We will follow the seven steps as you have learned about earlier in the PBL introduction course but with a bit of a twist. After the discussion of the scenario there will be a blackboard lecture of the more traditional type giving the main line of reasoning behind the subject of the cycle. The main part of each cycle is the self study in order to reach the goals discussed during the main discussion. Each cycle will also come with a set of problems to solve.

The five cycles cover different aspects of field theory

- Why Field Theory and the Klein-Gordon Field
- The Dirac Field and Fermions
- Interacting Fields and Feynman Diagrams
- Elementary Processes of Quantum Electrodynamics
- An introduction to Higher Order Corrections

2.2 Why this order in the cycles?

There are many ways to study Field Theory. The traditional way was to study the Dirac equation and then introduce second quantization. Another method is to simply postulate the path integral for field thory as the basic quantity and then take it from there. These approaches can be found in the various books given in he literature list. The route we follow here is a different one.

2.2.1 Cycle 1: A Noninteracting Scalar Field

The first cycle shows that no simple generalization of quantum mechanics can be made relativistically causal. Any relativistic generalization of the simple Schrödinger equation will have the same type of problems with causality. This hows that a much more drastic generalization of quantum mechanics is needed. This generalization is field theory and we will start by treating the quantum mechanics of this system, we circumvent the causality problem by enormously increasing the number of degrees of freedom, in fact instead of one particle with three spatial degrees of freedom we introduce a system with an infinite number of degrees of freedom. The first cycle is concerned with the simplest such system, a noninteracting scalar field, the Klein-Gordon field. In the end, we will see in detail how the causality problem is solved and how we obtain anti-particles.

2.2.2 Cycle 2: A noninteracting Fermion Field

This cycle covers several new aspects. There are more possibilities to write invariant wave equations than the simple Klein-Gordon equation and field introduced in the first cycle. he next simplest option is the one discovered by Dirac and is the main subect of this cycle. We will try to quantize this field with spin in the same way as the previous cycle and discover this leads to physical impossibilities. These require that we introduce fermions, particles that obey Fermi-Dirac statistics and the Pauli exclusion principle. The physics behind this problem is the cause of the spin-statistics theorem, an extremely strong consequence of field theory, integer spin particles are bosons, half-integerspin particles are fermions. Fermions also behave very nontrivial under a set of extra discrete symmetries, C, P and T. Again here some indication of an extremely general result exists, The combination CPT is a symmetry of any field theory.

2.2.3 Cycle 3: Interaction and Feynman Rules

The first two cycles introduced a lot of new concepts but the actual systems discussed are simple, sets of noninteracting particles. In this cycle we introduce interactions.The problem is that that very few field theories can be exactly solved and no physically relevant one has been solved exactly. We are thus forced to use approximate methods known as perturbation theory. Standard¡quantum mechanics perturbation theory can be used but is extremely cumbersome due to the infinite number of degrees of freedom and the fact that relativistic invariance is broken at all intermediate steps. Luckily, Feynman discovered a simpler way to perform these calculations and it is this method we will develop in this cycle. First for a very simple quantity, a two-point function, where all the steps can be explicitly followed, leading to Feynman rules for the two-point function and other correlators. From this we introducethe S-matrix and the concept of cross-section and the generalization of Feynman rules to this case.

2.2.4 Cycle 4: QED and applications

Finally, physics, not only formalism. This cycle serves to discover the basic physical processes in Quantum ElectroDynamics (QED) and their compariso with experiment. It also discusses how many of the features present in the processes calculated depend o a few basic physical aspects as momentum and angular momentum conservation.

2.2.5 Cycle 5: Divergences and how do we treat them

After all the work, we have really only calculated some results to lowest order in the perturbation theory expansion. This cycle willtake us beyond that and introduce is to two main new features that show up at higher orders. Infrared and ultraviolet divergences as well as new physical results. We first isolate a piece that can be calculated and obtain one of the first higher order corrections known in field theory, the anomalous magnetic moment. After that success we go back and analyze the remaining pieces and find trouble all over the place, both at the high and low end of the possible intermediate momenta. We show how the problem at the low end is solved by actually calculating only physically observable quantities which always includesoft Bremsstrahlung and indicate how the problem at the high end is solved by the concept of renormalization. Details of the latter are hwoever not part of this course.

2.3 Evaluation

The problem sets together with the final discussions and evaluation of each cycle are the major part of the assessment of this course but there will also be a short oral exam to test understanding of the principles. A list of typical questions for the latter is included.

3 Literature

The main literature is the book “Quantum Field Theory” by Peskin and Schroeder. The course will cover the first six chapters of this book, roughly one third of its contents. There are many good books and other sources on Quantum Field Theory. These will be given with the information for the first cycle.

Cycle 1

A problem with relativity

Jane and John have just learned quantum mechanics. So they calculate the probability for a free particle that at time $t = 0$ is at $\vec{x} = \vec{o}$ to be at $\vec{x} = \vec{y}$ at time t.

$$\begin{aligned} U(\vec{y}, t; \vec{o}, 0) &= \langle \vec{y}, t | e^{-iHt} | \vec{o}, 0 \rangle \\ &= \langle \vec{y}, t | \int \frac{d^3p}{(2\pi)^3} \, |\vec{p}\rangle \langle \vec{p}| e^{-i\frac{\vec{p}^2}{2m}t} |\vec{p}\rangle \langle \vec{p} | \vec{o}, 0 \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} e^{-i\frac{\vec{p}^2}{2m}t} e^{i\vec{p}\cdot\vec{y}} \\ &= \left(\frac{m}{2\pi i t}\right)^{\frac{3}{2}} e^{im\vec{y}^2/(2t)} \,. \end{aligned} \tag{1}$$

After looking at it for a while, Jane says: "But this goes against special relativity, the particle has a chance to be everywhere after any time t. Is this really true?"

Goals and Literature List for Cycle 1 of Relativistic Quantum Mechanics and Introductory Quantum Field Theory

Literature

The course is essentially an introduction to Quantum Field Theory and will cover in detail the first five chapters of

Michael E. Peskin and Daniel V. Schroeder, *An Introduction to Quantum Field Theory* (1995)

as well as the basic ideas of Chapter 6.

Other useful literature is basically any book with *Quantum Field Theory* in the title. Introductory level books covering basically what will be in this course are:

F. Mandl and G. Shaw, *Quantum field theory* (1993) L.H. Ryder, *Quantum field theory* (1996)

Some of the more philosophical backgrounds of Field theory can be found in the review article Frank Wilczek, Quantum Field Theory published in Rev.Mod.Phys. 71 (1999) S85-S95 or hep-th/9803075. A shorter discussion by Steven Weinberg is hep-th/9702027.

A book which provides insights in a somewhat different fashion is

H. Lipkin, *Quantum Mechanics: new approaches to selected topics*

Some background material relevant for this week's cycle is Ch. 13.

There is even a good (but rather advanced) Quantum Field Theory book on the web (download the ps or pdf file, don't try printing it out it has very many pages)

W. Siegel, *Fields*, hep-th/9912205

Goals

- Why Fields
- Classical Field Theory
- Free Fields via Harmonic Oscillators
- Moving in space-time: Causality
- Propagators and Sources

Exercises for Cycle 1 of Relativistic Quantum Mechanics and Introductory Quantum Field Theory

Do exercise 2.2 in Peskin and Schroeder (pages 33 and 34).

Cycle 2

Other solutions to the relativity problem?

John and Jane have now realized that quantizing the bosonic field solves the causality problem. But they now wonder whether there are other solutions.

Jane: "We figured out that the bosons had intrinsic angular momentum, i.e. spin, zero."

John: "But we learned that an electron has spin 1/2 and a photon has spin one. How should we treat this?"

Jane: "Well, what do we know about Lorentz transformations, and how should we try to get at other possible solutions? Besides, the Pauli principle should somehow come in as well for the electron, shouldn't it."

Goals, Literature and Exercise List for Cycle 2 of Relativistic Quantum Mechanics and Introductory Quantum Field Theory

Some points from last time

- Negative energy solutions: led to antiparticles
- Relativity: causality, negative energies are always possible, Lorentz transformations
- Fields are important: a wave equation alone is not causal, even if it is relativistically invariant.

Literature

This cycle of the introduction to Quantum Field Theory will cover the third chapter of

Michael E. Peskin and Daniel V. Schroeder, *An Introduction to Quantum Field Theory* (1995)

The contents are covered in most quantum field theory books in a chapter called something similar to *quantizing the Dirac equation* or *Fermions*

Goals

- Wave Equations and Lorentz Invariance
- Dirac Equation and its Free Particle Solutions; reducability of the Lorentz-group.
- Quantization of the Dirac field, why are they fermions
- Dirac propagator
- Dirac Field Bilinears and Discrete Symmetries
- It might be useful if you read 4.1 before the next cycle.

Exercises

- 3.2
- 3.4

(a-d) Some useful, equations are (prove them if you use them) $\sigma^2\vec{\sigma}^*\sigma^2 = -\vec{\sigma}$ and thus $\sigma^2\sigma^{\mu *}\sigma^2 = \bar{\sigma}^\mu$; $\sigma^{\mu *} = \sigma^{\mu T}$; $\chi_1^T\sigma^2\chi_2 = \chi_2^T\sigma^2\chi_1$ (fermions)

(b) There is an independent Grassmann variable for each component of the spinors at each point of spacetime as well as for the complex conjugates.

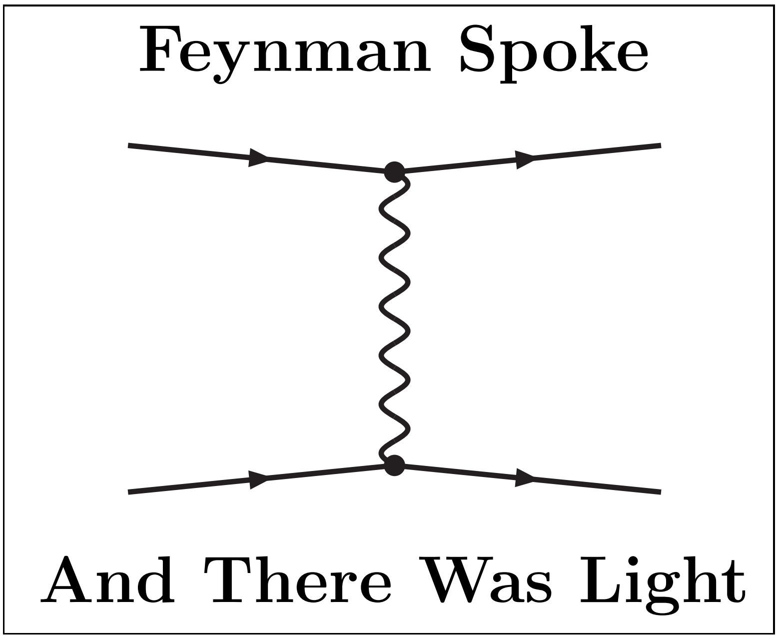
(e) Hint: The Majorana field can also be written with a four-spinor which then satisfies $\psi = -i\gamma^2\psi^*$. This gives you relations between the creation/annihilation operators for a Dirac fermion.

- 3.5 This exercise is difficult, you need to use a lot of the relations given in the previous exercise. The main reason for assigning is that you will encounter supersymmetry in some of the other courses. So, don't despair if you can't do it. It is not required for this course.

Cycle 2

A bit more action

Jane and John see a poster on the wall depicting something that looks like



Hmm, what is that all about, John wonders. Jane says, well those arrows look like how they drew the Dirac fermion propagators in the Quantum Field Theory course. Let's analyze that picture in more detail.

Goals, Literature and Exercise List for Cycle 3 of Relativistic Quantum Mechanics and Introductory Quantum Field Theory

Literature

This cycle of the introduction to Quantum Field Theory will cover most of the fourth chapter of

Michael E. Peskin and Daniel V. Schroeder, *An Introduction to Quantum Field Theory* (1995)

The contents are covered in most quantum field theory books in a chapter called something similar to *Perturbation Theory* or *Elementary Processes*.

Goals

- Green functions or Correlation functions
- Perturbation Theory: philosophy and expansion of correlation function
- Wick's theorem and Feynman Diagrams
- Cross Sections and the S Matrix
- Feynman rules: computing S Matrix Elements from Feynman Diagrams
- Idem but for Fermions
- Section 4.8 will be included in the next cycle

Exercises

- Note: There is one point which was not mentioned (explicitly) in the book (at least I did not find it) but is needed to get the correct answer. When you have identical particle in the final state, doing the integration over all of phase-space leads to double counting. For n identical particles in the final state, you need to divide the cross-section or decay rate by $n!$. This because if particle 1 and 2 are identical, when you interchange their momenta p_1 and p_2 you have the same final state which should only be counted once, not twice.
- 4.2

- 4.3
  - Before you start quantizing a spontaneously broken theory, you have to shift the field so that the new field has no vacuum expectation value. The vacuum expectation values are determined by finding the minimum of the field. This is as was done in e.g. FYS230 theoretical particle physics.
  - For the last part of 4.3(c), you have to use extensively the fact that $p_1 + p_2 = p_3 + p_4$. This together with $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$ allows you to prove that it vanishes at $\mathcal{O}(p^2)$. E.g. I used $p_1.p_2 - p_1.p_3 - p_1.p_4 - p_2.p_3 - p_2.p_4 + p_3.p_4 = 0$.
  - For 4.3(d), first prove that the ϕ^i for $i = 1, \ldots, N-1$ have now zero vacuum expectation value. Then solve the remaining equation for ϕ^N as a perturbation series in a, i.e. keep only terms with at most one factor of a. It can be done fully analytical in general, the full analytic solution to a cubic equation is known, but it will be very long.
- Note: 4.4 will be on the next cycle

Cycle 2

Something measured

John and Jane visit the control room of the Mark-I experiment at the electron-positron collider SPEAR at SLAC in Stanford in California. They see printouts of events with a muon-antimuon pair and events with a pair of photons lying around.

Goals, Literature and Exercise List for Cycle 4 of Relativistic Quantum Mechanics and Introductory Quantum Field Theory

Literature

This cycle of the introduction to Quantum Field Theory will cover the last section of the fourth chapter and the entire fifth chapter of

Michael E. Peskin and Daniel V. Schroeder, *An Introduction to Quantum Field Theory* (1995)

The contents are covered in most quantum field theory books in a chapter called something similar to *Elementary Processes*.

Goals

- Quantum Electrodynamics: Feynman rules
- $e^+e^- \to \mu^+\mu^-$ as the first process
- R as one of the main evidences for quarks
- $e^+e^- \to \mu^+\mu^-$: Helicity structure
- Production and decay of bound states
- $e^-\mu^- \to e^-\mu^-$ and crossing symmetry
- Compton scattering and the role of gauge invariance
- $e^+e^- \to \gamma\gamma$

Exercises

- 4.4 Note: the Fourier transform is the same as for the Yukawa potential and only afterwards send the (photon) mass to zero. It can of course be taken from the relevant part of 4.8.
- 5.1 Note: When you send $m_\mu \to \infty$ the center of mass system and the lab system (i.e. muon at rest) coincide.
- 5.2 Note: don't forget that it is only asked for electrons with zero mass.

Cycle 2

Beyond tree

So quantum field theory works fine, we calculated a lot of processes and found that they agree well with experiment. However, on closer examination it was noticed that all the references were to papers in the 1930s.

Jane remembered from browsing on the web that quantum field theory got a Nobel prize and John and Jane together looked up on `http://nobelprize.org`

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Nobelprize 1965

for their fundamental work in quantum electrodynamics, with
deep-ploughing consequences for the physics of elementary particles

Sin-Itiro Tomonaga      Julian Schwinger      Richard P. Feynman
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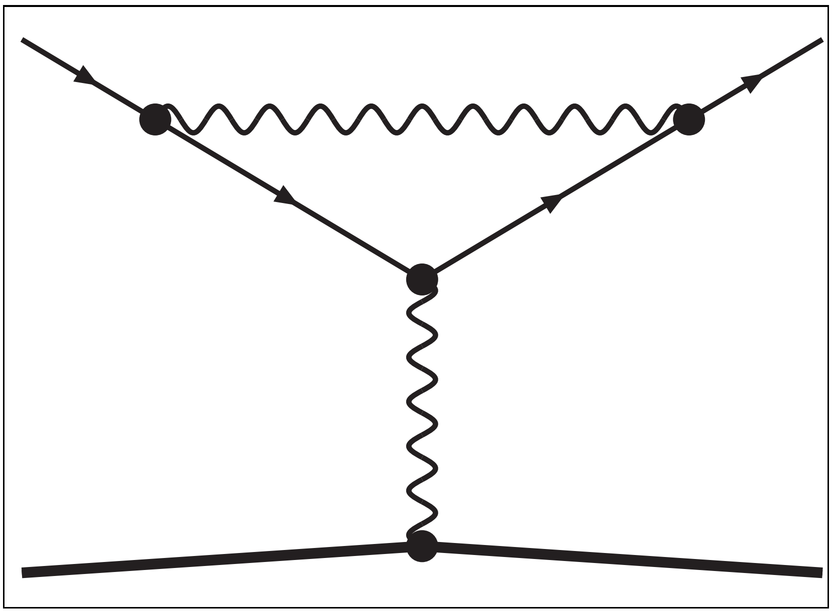
Reading through the Nobel lectures they find lots of words like

- backreaction of the field
- divergences
- radiative corrections
- anomalous magnetic moment
- Lamb shift

In a more popular description they also find the picture:



Goals, Literature and Exercise List for Cycle 5 of Relativistic Quantum Mechanics and Introductory Quantum Field Theory

Literature

This cycle of the introduction to Quantum Field Theory will cover the sixth chapter of

Michael E. Peskin and Daniel V. Schroeder, *An Introduction to Quantum Field Theory* (1995)

The contents are covered in most quantum field theory books in a chapter called something similar to *Radiative corrections*.

Goals

- Which diagrams appear?
- Section 6.2: Vertex general structure: what are form factors good for
- Section 6.3: Evaluation including Feynman parameters and momentum integrals
- A real QED prediction: g-2
- renormalization
- Section 6.1 Soft Bremsstrahlung
- Section 6.4: the infrared divergence from the electron vertex
- Section 6.5: the general infrared cancellations (only principles, no calculations needed)

Exercises

6.1 Note: you might find that judicious use of the Gordon identity simplifies matter considerably, alternatively use a program to do the gamma algebra for you.

Oral Exam Questions

Typical oral exam questions for QFT I

Chapter 2

- Explain why we want to introduce fields
- What is and sketch the derivation of Noether's theorem
- What is the relation between fields, canonical momentum, annihilation and creation operators.
- Sketch the derivation of the Klein-Gordon propagator
- How is the causality problem solved in QFT?

Chapter 3

- What is meant by Lorentz invariance in wave equations
- Give some examples of Lorentz invariant wave equations
- What are gamma matrices and what is the Dirac equation
- Why is a Dirac field a fermion (i.e. what is basic physical reason for the spin-statistics theorem of QFT?)
- What is/sketch the derivation of the Dirac propagator
- What are C, P and T? Is there something special about CPT?

Chapter 4

- The evolution operator $U(t_1, t_2)$ plays an important role in this chapter, sketch how we can obtain it in terms of H_I.
- $|\Omega\rangle$ and $|0\rangle$ are related via the time evolution; how does this come about?
- What is Wick's theorem and why do we use it?
- Sketch the cancellation of disconnected diagrams (in outline only).
- What is the S-matrix?
- What are Feynman rules and can you give some examples (e.g. for QED)

Chapter 5

- What is a cross-section
- What is crossing symmetry, can you argue why it should be valid
- What steps are involved in calculating a cross-section

Chapter 6

- What is a form-factor and why is it a useful concept?
- What is and how do we deal with the UV divergence in the vertex diagram.
- What is the infrared divergence?
- Describe the physical solution to the infrared divergence problem.