Four Lectures in Particle Dynamics

Lecture 2: The Linear Solution

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The Hamilton Equations

In the first lecture we wrote the Hamilton equations in a compact vectorial format as:

$$\frac{d\vec{v}}{dt} = S\vec{\nabla}^T H \tag{1}$$

where $\vec{v}$ is a vector for both coordinates and momenta; $\vec{\nabla}^T H$ is the gradient of the Hamiltonian calculated with respect to the coordinates and momenta and it is transposed because the gradient is generally defined as a row vector; S is the matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

The Linear Solution

If the Hamiltonian H is an homogeneous polynomial of degree 2 in $\vec{p}$ and $\vec{q}$ then we have

$$\vec{\nabla}^T H = W\vec{v} \tag{2}$$

where W is a matrix. As a consequence, the equations of Hamilton are

$$\frac{d\vec{v}}{dt} = S\vec{\nabla}^T H = SW\vec{v}. \tag{3}$$

If we call $\vec{v}_0$ the vector of initial conditions

$$q_1(0), \ldots, q_n(0), p_1(0), \ldots, p_n(0)$$

then the solution of Eq. (3) is

$$\vec{v}(t) = e^{tSW}\vec{v}_0. \tag{4}$$

The quantity e^{tSW} is a matrix calculated using the Euler formula for the exponential

$$M(t) = e^{tSW} = \sum_{n=0}^{\infty} \frac{t^n}{n!} (SW)^n \,. \tag{5}$$

$M(t)$ is called the transfer matrix of the coordinates and momenta, which is symplectic because it is the solution of the Hamilton equations.

Exercise: the classical harmonic oscillator

The Hamiltonian of a classical harmonic oscillator in one dimension is

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2. \tag{6}$$

Using the formalism of Eq. (4) find the transfer matrix $M(t)$ and verify that it is symplectic.

Linear solution of electromagnetic Hamiltonian

The general Hamiltonian for a particle traveling in the electromagnetic field, in the coordinates of a reference particle is

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\delta + \frac{1}{\beta_0} - \frac{q_c \phi}{c p_0}\right)^2 - (p_x - a_x)^2 - (p_y - a_y)^2 - \frac{1}{\beta_0^2 \gamma_0^2}} - a_z \quad (7)$$

that in general is not quadratic in coordinates and momenta. As a consequence, the way to use the solution (4) is to calculate the Taylor expansion of the Hamiltonian and truncate it at the second order. If there are no first order terms, then the linear solution can be applied.

As an example we will calculate the propagation of a particle in a drift space, without any force applied, i.e. $\vec{a} = 0$ and $\phi = 0$.

Drifting particle in empty space

The Hamiltonian is

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\delta + \frac{1}{\beta_0}\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} \tag{8}$$

and the Taylor expansion is

$$H = \cancel{-1} + \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{\delta^2}{2\beta_0^2\gamma_0^2} + \cancel{\mathcal{O}(3)} \tag{9}$$

where the constant term is nonphysical (all the equations involve derivatives of H) and we want to truncate the expansion at the order 2. Now that we have a quadratic polynomial we can apply the solution (4) considering that the free parameter of this Hamiltonian is no longer the time t but the distance traveled by the particle.

Drifting particle in empty space

We first calculate the matrix W as

$$\vec{\nabla}^T H = W\vec{v} \tag{10}$$

$$\begin{pmatrix} \frac{\partial}{\partial x}\left(\frac{p_x^2}{2}+\frac{p_y^2}{2}+\frac{\delta^2}{2\beta_0^2\gamma_0^2}\right) \\ \frac{\partial}{\partial y}\left(\frac{p_x^2}{2}+\frac{p_y^2}{2}+\frac{\delta^2}{2\beta_0^2\gamma_0^2}\right) \\ \frac{\partial}{\partial z}\left(\frac{p_x^2}{2}+\frac{p_y^2}{2}+\frac{\delta^2}{2\beta_0^2\gamma_0^2}\right) \\ \frac{\partial}{\partial p_x}\left(\frac{p_x^2}{2}+\frac{p_y^2}{2}+\frac{\delta^2}{2\beta_0^2\gamma_0^2}\right) \\ \frac{\partial}{\partial p_y}\left(\frac{p_x^2}{2}+\frac{p_y^2}{2}+\frac{\delta^2}{2\beta_0^2\gamma_0^2}\right) \\ \frac{\partial}{\partial \delta}\left(\frac{p_x^2}{2}+\frac{p_y^2}{2}+\frac{\delta^2}{2\beta_0^2\gamma_0^2}\right) \end{pmatrix} = W \begin{pmatrix} x \\ y \\ z \\ p_x \\ p_y \\ \delta \end{pmatrix} \tag{11}$$

Drifting particle in empty space

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ p_x \\ p_y \\ \frac{\delta}{\beta_0^2\gamma_0^2} \end{pmatrix} = W \begin{pmatrix} x \\ y \\ z \\ p_x \\ p_y \\ \delta \end{pmatrix} \tag{12}$$

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\beta_0^2\gamma_0^2} \end{pmatrix} \tag{13}$$

Drifting particle in empty space

As said the time variable is here substituted with the traveled space. So for a drift of length L the transfer map $M(L) = e^{LSW}$ is

$$\exp\left[L\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\beta_0^2\gamma_0^2} \end{pmatrix}\right] \tag{14}$$

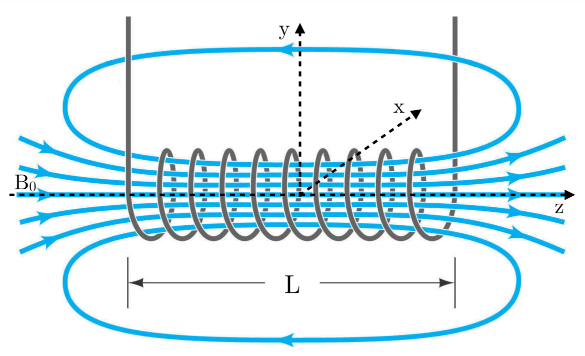
Drifting particle in empty space

$$M(L) = \exp \begin{pmatrix} 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{15}$$

$$= \begin{pmatrix} 1 & 0 & 0 & L & 0 & 0 \\ 0 & 1 & 0 & 0 & L & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{16}$$

Exercise: the solenoid

Apply the same procedure used for the drift to calculate the transfer matrix of an ideal solenoid of length L where the magnetic field is constant along the z axis and zero in the transversal plane $\vec{B} = (0, 0, B_0)$.



Multi particles

So far we learned how to track one particle through an electromagnetic field represented by a quadratic Hamiltonian. In a particle accelerator a bunch of particle can be composed by several billions of particles. Of course, once we have the transfer matrix M we can multiply the billions vectors $\vec{v}$, but this can be a very slow process. It does not exists a general way to transport the set of particles in a single calculation but we can define some statistical properties of the beam, such as the mean of positions and momenta and their standard deviations, and track them applying the theory described in [1] and expanded in [2], [3] and Chapter 4 of [4].

Multi particles

Here we will only give an idea of how this theory works. To simplify the discussion we will study the beam in one dimension $\vec{v} = (x, p_x)$. Given N particles running over an index i, we define the Σ matrix as

$$\begin{aligned}
\Sigma &= \left\langle (\vec{v}_i - \langle\vec{v}_i\rangle)(\vec{v}_i - \langle\vec{v}_i\rangle)^T \right\rangle \\
&= \frac{1}{N}\sum_i^N \left[\vec{v}_i - \frac{1}{N}\sum_i^N \vec{v}_i\right]\left[\vec{v}_i^T - \frac{1}{N}\sum_i^N \vec{v}_i^T\right] \\
&= \frac{1}{N}\sum_i^N \left[\begin{pmatrix} x \\ p_x \end{pmatrix}_i - \frac{1}{N}\sum_i^N \begin{pmatrix} x \\ p_x \end{pmatrix}_i\right]\left[\begin{pmatrix} x & p_x \end{pmatrix}_i - \frac{1}{N}\sum_i^N \begin{pmatrix} x & p_x \end{pmatrix}_i\right] \\
&= \frac{1}{N}\sum_i^N \left[\begin{pmatrix} x_i - \bar{x} \\ p_{x\,i} - \bar{p_x} \end{pmatrix}\right]\left[\begin{pmatrix} x_i - \bar{x} & p_{x\,i} - \bar{p}_x \end{pmatrix}\right] \\
&= \frac{1}{N}\sum_i^N \left[\begin{pmatrix} (x_i - \bar{x})^2 & (x_i - \bar{x})(p_{x\,i} - \bar{p}_x) \\ (x_i - \bar{x})(p_{x\,i} - \bar{p}_x) & (p_{x\,i} - \bar{p}_x)^2 \end{pmatrix}\right] \\
&= \begin{pmatrix} \sigma_x^2 & \sigma_{xp_x} \\ \sigma_{xp_x} & \sigma_{p_x}^2 \end{pmatrix}.
\end{aligned} \tag{17}$$

Multi particles

We can now prove that multiplying the transfer matrix M of a single particle left and right (transposed) for the Σ matrix, we have the Σ matrix after the electromagnetic element.

$$
\begin{aligned}
M\Sigma M^T &= M\frac{1}{N}\sum_i^N\left[\vec{v}_i - \frac{1}{N}\sum_i^N \vec{v}_i\right]\left[\vec{v}_i^T - \frac{1}{N}\sum_i^N \vec{v}_i^T\right]M^T \\
&= \frac{1}{N}\sum_i^N\left[M\vec{v}_i - \frac{1}{N}\sum_i^N M\vec{v}_i\right]\left[\vec{v}_i^T M^T - \frac{1}{N}\sum_i^N \vec{v}_i^T M^T\right] \\
&= \Sigma_L. \qquad (18)
\end{aligned}
$$

This proof works because the matrix M can be transported inside the sum and this is due to the linear properties of the matrix. It is clear that Σ is only one of the several statistical properties of the beam that can be transported with the same formalism of the single particle, for example the average is another one.

The three dimensional formalism

The Σ matrix has three degree of freedom because two entries are identical, so the equation $\Sigma_L = M\Sigma_0 M^T$ imposes three conditions

$$\sigma^2_{x\,L} = m^2_{11}\sigma^2_{x\,0} + 2m_{11}m_{12}\sigma_{xp_x\,0} + m^2_{12}\sigma^2_{p_x\,0} \tag{19}$$

$$\sigma_{xp_x\,L} = m_{11}m_{21}\sigma^2_{x\,0} + (m_{11}m_{22} + m_{12}m_{21})\sigma_{xp_x\,0} + m_{12}m_{22}\sigma^2_{p_x\,0} \tag{20}$$

$$\sigma^2_{p_x\,L} = m^2_{21}\sigma^2_{x\,0} + 2m_{21}m_{22}\sigma_{xp_x\,0} + m^2_{22}\sigma^2_{p_x\,0}. \tag{21}$$

or, in a compact form

$$\begin{pmatrix} \sigma^2_x \\ \sigma_{xp_x} \\ \sigma^2_{p_x} \end{pmatrix}_L = T \begin{pmatrix} \sigma^2_x \\ \sigma_{xp_x} \\ \sigma^2_{p_x} \end{pmatrix}_0 \tag{22}$$

with the matrix T given by

$$T = \begin{pmatrix} m_{11}^2 & 2m_{11}m_{12} & m_{12}^2 \\ m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & m_{12}m_{22} \\ m_{21}^2 & 2m_{21}m_{22} & m_{22}^2 \end{pmatrix}. \tag{23}$$

We have now the recipe to transport our statistical quantity of the beam from the Hamiltonian: first we solve for the single particle calculating the matrix M, then we generate the matrix T and we use it to transport the beam distribution.

Computational Exercise: the beam in the quadrupole

A quadrupolar magnet in one dimension has an Hamiltonian

$$H = \frac{p_x^2}{2} + k^2 \frac{x^2}{2}. \tag{24}$$

Generate 1000 random particles in one dimension (x, p_x) with $0 \leq x \leq 5e-3$ m and $0 \leq p_x \leq 5e-6$ and transport them with a quadrupole of length $L = 1$ m and k=0.05 m^{-1}. Evaluate the standard deviation of the positions and momenta before and after the transport of the particles and compare with the standard deviation transported with the T matrix. The two methods must give the same result.

Twiss functions

We close this lecture defining the functions of Twiss (first introduced by the Richard Q. Twiss) that are largely used in the formalism of the beam physics. These functions are α, β and γ defined as

$$\sigma_x^2 = \beta\epsilon; \quad \sigma_{xp_x} = -\alpha\epsilon; \quad \sigma_{p_x}^2 = \gamma\epsilon \tag{25}$$

The three functions are not independent, two equations hold:

$$\alpha = -\frac{1}{2}\frac{d\beta(s)}{ds}; \quad \gamma = \frac{1+\alpha^2}{\beta}; \quad \epsilon = \sqrt{\sigma_x^2\sigma_{p_x}^2 - \sigma_{xp_x}^2} \tag{26}$$

where s is the evolution parameter (time or position depending on the independent variable used). ϵ is called the emittance of the beam and it is proportional to the volume of the phase space as $V = \pi\epsilon$. Clearly it is an invariant because of the Liouville's theorem.

Twiss functions

This formalism is very useful in a circular accelerator. Under periodic conditions (the force repeats after a certain time) the solution of the Hamilton equations can be expressed in a unique way as periodic functions (This is the content of the Floquet's theorem). Then, in the special case of periodic forces, such as in a ring, the α and β functions are entirely determined by the force of the accelerator and are intrinsic to properties of the machine. Each particle will move in the accelerator oscillating with the β function around the center of the accelerator. The only free parameters are the amplitude and the phase of the oscillation of the particle. In other words, in a periodic machine, for each point and for each particle with index i holds the equation

$$\gamma x_i^2 + 2\alpha x_i p_{xi} + \beta p_{xi}^2 = \epsilon_i^2. \tag{27}$$

Twiss functions

In each point of a periodic machine with linear forces, each particle paints an ellipse in the phase space. The quantity ϵ_i is called the emittance of the particle or the invariant of Courant-Snyder. $\pi\epsilon_i$ is the volume of the phase space painted by the single particle in any accelerator's point.

It is remarkable that the Twiss functions describe, at the same time, the motion of a single particle and the motion of the bunch. This is related to the linear description that is valid for one or many particles at the same time.

Solutions to proposed exercises.

Solution: the classical harmonic oscillator

The Hamiltonian of a classical harmonic oscillator in one dimension is

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2. \tag{28}$$

Using the formalism of Eq. (4) find the transfer matrix $M(t)$ and verify that it is symplectic.

Solution: the classical harmonic oscillator

We first calculate the matrix W as

$$\begin{aligned}
\vec{\nabla}^T H &= W\vec{v} && (29)\\
\begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{pmatrix} &= W \begin{pmatrix} x \\ p \end{pmatrix} && (30)\\
\begin{pmatrix} \frac{\partial}{\partial x}\left(\frac{p^2}{2m} + \frac{1}{2}kx^2\right) \\ \frac{\partial}{\partial p}\left(\frac{p^2}{2m} + \frac{1}{2}kx^2\right) \end{pmatrix} &= W \begin{pmatrix} x \\ p \end{pmatrix} && (31)\\
\begin{pmatrix} kx \\ \frac{p}{m} \end{pmatrix} &= W \begin{pmatrix} x \\ p \end{pmatrix} && (32)\\
W &= \begin{pmatrix} k & 0 \\ 0 & \frac{1}{m} \end{pmatrix}. && (33)
\end{aligned}$$

Solution: the classical harmonic oscillator

We then calculate

$$tSW = t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & \frac{1}{m} \end{pmatrix} = \begin{pmatrix} 0 & \frac{t}{m} \\ -tk & 0 \end{pmatrix} \tag{34}$$

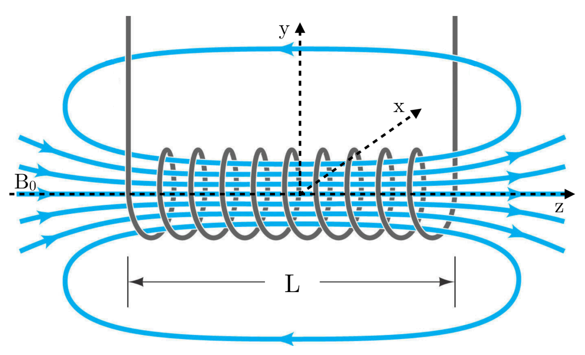
and finally

$$M(t) = e^{tSW} = \exp\left[\begin{pmatrix} 0 & \frac{t}{m} \\ -tk & 0 \end{pmatrix}\right] \tag{35}$$

$$= \begin{pmatrix} \cos\left(\sqrt{\frac{k}{m}}t\right) & \frac{1}{\sqrt{km}}\sin\left(\sqrt{\frac{k}{m}}t\right) \\ -\sqrt{km}\sin\left(\sqrt{\frac{k}{m}}t\right) & \cos\left(\sqrt{\frac{k}{m}}t\right) \end{pmatrix}. \tag{36}$$

The proof of symplecticity is straightforward because in 2 dimensions the condition is equivalent to have the determinant equal to 1 and this is evident from the matrix M.

Apply the same procedure used for the drift to calculate the transfer matrix of an ideal solenoid of length L where the magnetic field is constant along the z axis and zero in the transversal plane $\vec{B} = (0, 0, B_0)$.



Solution: the solenoid

The electric field of the solenoid is zero, so $\phi = 0$. The magnetic potential has to be such that $\vec{B} = \vec{\nabla} \times \vec{A}$. The choiche of $\vec{A}$ is not unique (this freedom is called gauge symmetry), but we will choose the convenient potential

$$\vec{A} = \left(-\frac{B_0}{2} y, \frac{B_0}{2} x, 0 \right). \tag{37}$$

We have also to recall that the coordinate transformation applied in our Hamiltonian transformed the vector potential $\vec{A}$ into $\vec{a} = \frac{q}{p_0} \vec{A}$. Calling $k = \frac{q}{p_0} \frac{B_0}{2}$ we finally have

$$\vec{a} = (-ky, kx, 0) \tag{38}$$

and the Hamiltonian is

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\delta + \frac{1}{\beta_0} \right)^2 - (p_x + ky)^2 - (p_y - kx)^2 - \frac{1}{\beta_0^2 \gamma_0^2}}. \tag{39}$$

Solution: the solenoid

The paraxial approximation (second order expansion) is

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{\delta^2}{2\beta_0^2\gamma_0^2} + \frac{k^2}{2}x^2 + \frac{k^2}{2}y^2 - kxp_y + kyp_x \tag{40}$$

to find the matrix W we calculate the derivatives of the Hamiltonian

$$W \begin{pmatrix} x \\ y \\ z \\ p_x \\ p_y \\ \delta \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial x} \\ \frac{\partial \mathcal{H}}{\partial y} \\ \frac{\partial \mathcal{H}}{\partial z} \\ \frac{\partial \mathcal{H}}{\partial p_x} \\ \frac{\partial \mathcal{H}}{\partial p_y} \\ \frac{\partial \mathcal{H}}{\partial \delta} \end{pmatrix} = \begin{pmatrix} k^2x - kp_y \\ k^2y + kp_x \\ 0 \\ ky + p_x \\ -kx + p_y \\ \frac{\delta}{\beta_0^2\gamma_0^2} \end{pmatrix} \tag{41}$$

Solution: the solenoid

$$W = \begin{pmatrix} k^2 & 0 & 0 & 0 & -k & 0 \\ 0 & k^2 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 1 & 0 & 0 \\ -k & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\beta_0^2\gamma_0^2} \end{pmatrix} \tag{42}$$

and we continue calculating SW as

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k^2 & 0 & 0 & 0 & -k & 0 \\ 0 & k^2 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 1 & 0 & 0 \\ -k & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\beta_0^2\gamma_0^2} \end{pmatrix} \tag{43}$$

Solution: the solenoid

and the transfert matrix is

$$M(k,L) = \exp \begin{pmatrix} 0 & kL & 0 & L & 0 & 0 \\ -kL & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{L}{\beta_0^2\gamma_0^2} \\ -k^2L & 0 & 0 & 0 & kL & 0 \\ 0 & -k^2L & 0 & -kL & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{44}$$

$$= \begin{pmatrix} \cos^2(Lk) & \frac{\sin(2Lk)}{2} & 0 & \frac{\sin(2Lk)}{2k} & \frac{\sin^2(Lk)}{k} & 0 \\ -\frac{\sin(2Lk)}{2} & \cos^2(Lk) & 0 & -\frac{\sin^2(Lk)}{k} & \frac{\sin(2Lk)}{2k} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{L}{\beta_0^2\gamma_0^2} \\ -\frac{k\sin(2Lk)}{2} & -k\sin^2(Lk) & 0 & \cos^2(Lk) & \frac{\sin(2Lk)}{2} & 0 \\ k\sin^2(Lk) & -\frac{k\sin(2Lk)}{2} & 0 & -\frac{\sin(2Lk)}{2} & \cos^2(Lk) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{45}$$

Solution: the beam in the quadrupole

A quadrupolar magnet in one dimension has Hamiltonian

$$H = \frac{p_x^2}{2} + k^2\frac{x^2}{2}. \tag{46}$$

Generate 1000 random particles in one dimension (x, p_x) with $0 \le x \le 5e-3$ m and $0 \le p_x \le 5e-6$ and transport them with a quadrupole of length $L = 1$ m and k=0.05 m^{-1}. Evaluate the standard deviation of the positions and momenta before and after the transport of the particles and compare with the standard deviation transported with the T matrix. The two methods must give the same result.

Solution: the beam in the quadrupole

The first step is to calculate the transfer matrix. This is exactly as the solution of the harmonic oscillator when the mass is equal to 1, the time is equal to L and with k^2 instead of k. Then the transfer matrix is

$$M(k, L) = \begin{pmatrix} \cos(kL) & \frac{1}{k}\sin(kL) \\ -k\sin(kL) & \cos(kL) \end{pmatrix}. \tag{47}$$

We have all the information required to solve this problem, so we can start to create our simulator and track the 1000 particles as requested. The solution here will be proposed using the programming language Python.

Solution: the beam in the quadrupole

We start with a code that generates the 1000 particles.

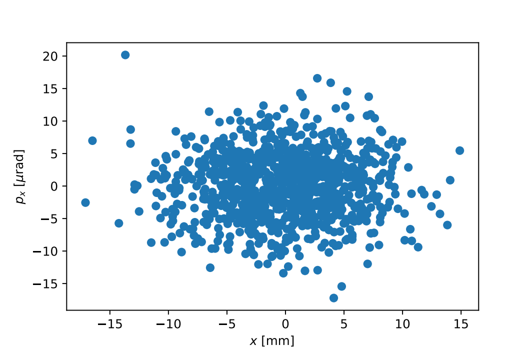
```
x_array = random.normal(0.0, 5e-3, size=1000)
px_array = random.normal(0, 5e-6, size=1000)

sigma_x = x_array.std()**2
sigma_px = px_array.std()**2
sigma_xpx = sum((x_array-x_array.mean())*(px_array-
    px_array.mean()))/len(x_array)
sigma_x, sigma_xpx, sigma_px

plt.xlabel('$x$ [mm]')
plt.ylabel('$p_x$ [$\mu$rad]')
plt.scatter(x_array/1e-3, px_array/1e-6) # I divide for
    the plot scale
plt.show()
```

Solution: the beam in the quadrupole

The code produces the following distribution and RMS:



$$
\begin{aligned}
\sigma_x^2 &= 2.41792614935e-05 \\
\sigma_{xp_x} &= 1.9160691948e-10 \\
\sigma_{p_x}^2 &= 2.51356321556e-11.
\end{aligned}
$$

Solution: the beam in the quadrupole

Then we create the matrix M and we transport the particles

```
k = 0.05
L = 1.0
M = matrix([[cos(k*L), sin(k*L)/k], [-k*sin(k*L), cos(k*L)]])
new_x_array = zeros(len(x_array))
new_px_array = zeros(len(px_array))
for i in range(len(x_array)):
    new_x_array[i] = (M*array([[x_array[i]], [px_array[i]]]))[0]
    new_px_array[i] = (M*array([[x_array[i]], [px_array[i]]]))[1]
```

Solution: the beam in the quadrupole

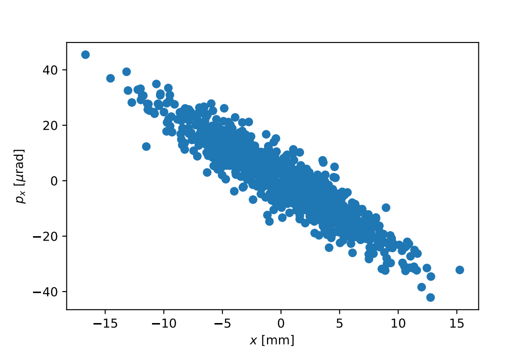
And we calculate the RMS

```
new_sigma_x = new_x_array.std()**2
new_sigma_px = new_px_array.std()**2
new_sigma_xpx = sum((new_x_array-new_x_array.mean())*(
    new_px_array-new_px_array.mean()))/len(new_x_array)
new_sigma_x, new_sigma_xpx, new_sigma_px

plt.xlabel('$x$ [mm]')
plt.ylabel('$p_x$ [$\mu$rad]')
plt.scatter(new_x_array/1e-3, new_px_array/1e-6) # I
    divide in the plot but the array are in meters and
    rad
plt.show()
print (new_sigma_x, new_sigma_xpx, new_sigma_px)
```

Solution: the beam in the quadrupole

The final distribution and RMS are:



$$
\begin{aligned}
\sigma_x^2 &= 2.41192713866e-05 \\
\sigma_{xp_x} &= -6.0131713729e-08 \\
\sigma_{p_x}^2 &= 1.75110899404e-10.
\end{aligned}
$$

Solution: the beam in the quadrupole

Now we compare with the results obtained with the T matrix

```
T = matrix([[M[0,0]**2, 2*M[0,0]*M[0,1], M[0,1]**2], [M[0,0]*M[1,0], M[0,0]*M[1,1]+M[0,1]*M[1,0], M[0,1]*M[1,1]], [M[1,0]**2, 2*M[1,0]*M[1,1], M[1,1]**2]])

sigma_array = array([[sigma_x], [sigma_xpx], [sigma_px]])
print('The initial array is')
print(sigma_array)
new_sigma_array = T*sigma_array
new_sigma_array
print('The final array is')
print(new_sigma_array)
```

Solution: the beam in the quadrupole

The initial values are

$$\begin{aligned} \sigma_x^2 &= 2.41792614935e-05 \\ \sigma_{xp_x} &= 1.9160691948e-10 \\ \sigma_{p_x}^2 &= 2.51356321556e-11. \end{aligned}$$

and the final values are

$$\begin{aligned} \sigma_x^2 &= 2.41192713866e-05 \\ \sigma_{xp_x} &= -6.0131713729e-08 \\ \sigma_{p_x}^2 &= 1.75110899404e-10. \end{aligned}$$

confirming that the theory is correct.

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