# Four Lectures in Particle Dynamics

## Lecture 4: The Symplectic Integrator

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# Lost of Symplecticity

When the Hamiltonian is not quadratic, therefore the solution is non-linear, we proposed a general solution based on the Lie transform. The transfer map can be written as

$$e^{-t:H:} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}(: H :)^n \tag{1}$$

and it is not in general possible to calculate it exactly. Moreover, the series expands in power of $t$ and not as a power of the coordinates and momenta, so our expansion can be a problem because we do not have the full control of the error that we are introducing with respect to the distance of the particle from the synchronous particle.

## Lost of Symplecticity

But the real physical problem is that the approximation of the Eq. (1) destroys the symplecticity. The effect is that the phase space can inflate or shrink artificially showing a variation of energy in the beam that does not correspond to the physical reality. The more non-linear elements we cross the worst can be the effect of the approximation. Is there a way to restore the symplecticity?

# A useful property of Lie transform

Let say for one moment that our Hamiltonian is a function only of one of the two canonical variables

$$H = H(p_x) \tag{2}$$

then when used as Lie operator on the canonical variables we see

$$:H:x = \cancel{\frac{\partial H}{\partial x}} \cancel{\frac{\partial x}{\partial p_x}} - \frac{\partial H}{\partial p_x}\frac{\partial x}{\partial x} = -\frac{\partial H}{\partial p_x} \tag{3}$$

$$:H:p_x = \cancel{\frac{\partial H}{\partial x}} \frac{\partial p_x}{\partial p_x} - \frac{\partial H}{\partial p_x}\cancel{\frac{\partial p_x}{\partial x}} = 0 \tag{4}$$

where the cancelled terms are the derivatives equal to zero.

# A useful property of Lie transform

The quantity $-\frac{\partial H}{\partial p_x}$ is only function of $p_x$, so if we now iterate the Lie operator we obtain

$$\begin{aligned}
(:H:)^2 x &= \; :H: \left(-\frac{\partial H}{\partial p_x}\right) \\
&= \cancel{\frac{\partial H}{\partial x}} \frac{\partial}{\partial p_x}\left(-\frac{\partial H}{\partial p_x}\right) - \frac{\partial H}{\partial p_x} \cancel{\frac{\partial}{\partial x}\left(-\frac{\partial H}{\partial p_x}\right)} \\
&= 0 \qquad (5)
\end{aligned}$$

This is an interesting result: if the Hamiltonian is a function only of one canonical variable we can fully calculate the Lie transform of the coordinates and momenta because the orders higher than the second are all zero. Because of the symmetry of the Hamilton equations the property is valid also if $H$ is just a function of $x$, we have only to swap equations (and sign!).

# A useful property of Lie transform

To summarize we discovered that when $H = H_x$ is a function only of $x$ we have

$$e^{:H_x:}x = x \tag{6}$$

$$e^{:H_x:}p_x = p_x + \frac{\partial H_x}{\partial x} \tag{7}$$

while when $H_{p_x}$ is a function only of $p_x$ we have

$$e^{:H_{p_x}:}x = x - \frac{\partial H_{p_x}}{\partial p_x} \tag{8}$$

$$e^{:H_{p_x}:}p_x = p_x \tag{9}$$

# Exercise: Lie operator iteration of single variable function

If $f(x)$ is a function only of $x$ and $g(p_x)$ is a function only of $p_x$ prove that

$$
\begin{aligned}
(:H_x:)^n f(x) &= 0 && (10)\\
(:H_x:)^n g(p_x) &= \left(\frac{\partial H_x}{\partial x}\right)^n \frac{\partial^n g(p_x)}{\partial p_x^n} && (11)\\
(:H_{p_x}:)^n f(x) &= (-1)^n \left(\frac{\partial H_{p_x}}{\partial p_x}\right)^n \frac{\partial^n f(x)}{\partial x^n} && (12)\\
(:H_{p_x}:)^n g(p_x) &= 0 && (13)
\end{aligned}
$$

# A useful property of Lie transform

The final step that we need to learn is how to calculate the iteration of one Hamiltonian in one variable on the Hamiltonian in the other variable.

$$e^{:H_{p_x}:}e^{:H_x:}x = e^{:H_{p_x}:}x = x - \frac{\partial H_{p_x}}{\partial p_x} \tag{14}$$

$$\begin{aligned} e^{:H_{p_x}:}e^{:H_x:}p_x &= e^{:H_{p_x}:}\left(p_x + \frac{\partial H_x}{\partial x}\right) \\ &= p_x + e^{:H_{p_x}:}\left(\frac{\partial H_x}{\partial x}\right). \end{aligned} \tag{15}$$

A similar result is valid also if we invert the order of the Lie transforms. The Eq. (15) is in the same form of the Eq. (12) and if $H_x$ vanish after a certain number of derivatives (for example if $H_x$ is polynomial), both the Eqs. (14) and (15) are finite and we can calculate them exactly.

# Restoring Symplecticity

The discussion we did so far is very interesting because the Hamiltonian is in general written as the sum of two components $H = T + V$ where the kinetic component $T$ is a function only of the momenta, while the potential $V$ is generally only function of the positions.

Now let us return to our equations of dynamics

$$\begin{aligned} \vec{v}(t) &= e^{-t:H:}v_0 = e^{-t:T+V:}v_0 \\ &\stackrel{?}{=} e^{-t:T:}e^{-t:V:}v_0. \end{aligned} \tag{16}$$

If the last equality is valid and if the n$^{\text{th}}$ derivative of $T$ and $V$ vanish, then we have a formula to calculate our dynamics without truncating the series, in other words symplectic!

# Restoring Symplecticity

Unfortunately, the life of the physicist is never easy and the equation

$$e^{-t:T+V:} \stackrel{?}{=} e^{-t:T:}e^{-t:V:} \tag{17}$$

is not valid. The right way to split the sum in the operator is using the Zassenhaus formula that is the dual application of the more famous Backer-Campbell-Hausdorff formula (see Chapter 10.1 [2])

$$e^{-t:T+V:} = e^{-t:T:}e^{-t:V:}e^{\frac{t^2}{2}:\{T,V\}:}e^{-\frac{t^3}{3}:\{V,\{T,V\}\}:}e^{-\frac{t^3}{6}:\{T,\{T,V\}\}:}\dots \tag{18}$$

where the brackets $\{T, V\}$ are the Poisson brackets.

# Exercise: second order integrator

Considering the Zassenhaus formula at the second order

$$e^{-t:T+V:} \approx e^{-t:T:} e^{-t:V:} e^{\frac{t^2}{2}:\{T,V\}:} \tag{19}$$

show that if the Hamiltonian is split as $\frac{1}{2}T + V + \frac{1}{2}T$ the second order cancels and the equation becomes

$$e^{-t:\frac{1}{2}T+V+\frac{1}{2}T:} \approx e^{-t:\frac{1}{2}T:} e^{-t:V:} e^{-t:\frac{1}{2}T:}. \tag{20}$$

Hints: the Poisson Brackets satisfy the following rules

$$\{A+B, C\} = \{A, C\} + \{B, C\} \tag{21}$$

$$\{A, B+C\} = \{A, B\} + \{A, C\} \tag{22}$$

$$\{k_1 A, k_2 B\} = k_1 k_2 \{A, B\} \tag{23}$$

$$\{A, A\} = 0 \tag{24}$$

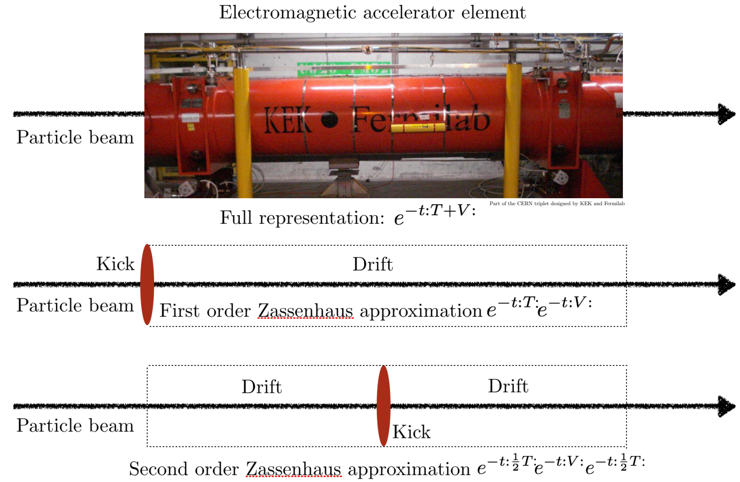
$$\{A, B\} = -\{B, A\} \tag{25}$$

# The Physical meaning

The equation (20) tells us that if we write the Hamiltonian as $T + V$ or if we write it as $\frac{1}{2}T + V + \frac{1}{2}T$ we have, of course, the same Hamiltonian, but a better approximation for the Zassenhaus formula because the second way cancels the second order terms of the formula. What is the physical meaning of this different way to write the Hamiltonian? We have to think that we are not splitting a simple function, but we are splitting the transformation through a purely kinetic element $T$, and a purely potential term $V$. The kinetic term is a drift (empty space with no forces) while the potential is an instantaneous kick where all the force of the element is "condensed".

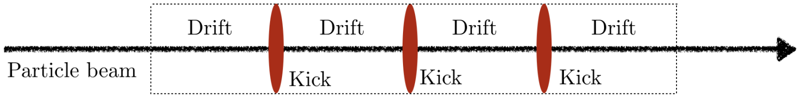
# The Physical meaning

Graphically we can see it as



# The Physical meaning

The technique to split the Hamiltonian in order to cancel the terms of the Zassenhaus formula is called Symplectic integration. We are tempted to think that the symmetry (kick in the middle point) is the reason why $\frac{1}{2}T + V + \frac{1}{2}T$ approximates better the dynamics compared to the $T + V$. If this is true for the second order, this is not true for the higher orders. If we place other two kicks in the middle of the drifts, this does not cancel the third order ($t^3$) in the Zassenhaus formula. In other words, the approximation in the figure is wrong.




$$e^{-t:\frac{1}{4}T:}e^{-t:\frac{1}{3}V:}e^{-t:\frac{1}{4}T:}e^{-t:\frac{1}{3}V:}e^{-t:\frac{1}{4}T:}e^{-t:\frac{1}{3}V:}e^{-t:\frac{1}{4}T:}$$

Wrong!

# The Yoshida Symplectic Integrator

The correct split that cancels the second, third and fourth, order terms in $t$ in the Zassenhaus formula is

$$H = d_1 T + c_1 V + d_2 T + c_2 V + d_2 T + c_1 V + d_1 T \tag{26}$$

where the coefficients are

$$d_1 = \frac{1}{12}\left(4 + 2\sqrt[3]{2} + \sqrt[3]{4}\right) \approx 0.6756 \tag{27}$$

$$d_2 = \frac{1}{2} - d_1 \approx -0.1756 \tag{28}$$

$$c_1 = 2d_1 \approx 1.3512 \tag{29}$$

$$c_2 = 1 - 4d_1 \approx -1.7024 \tag{30}$$

It is interesting to notice that the drift $d_2$ is negative. This corresponds to a kick applied backward and is not at all intuitive as a result.

## The Yoshida Symplectic Integrator

The coefficients (27-30) were calculated for the first time in [1]; the idea was to insert the Hamiltonian (26) into the Zassehaus formula in order to impose to the coefficients to cancel the high order terms. This technique can easily become very difficult to be extended because the calculations to perform for higher orders increase exponentially, so it is not possible to apply it for higher orders.

Luckily for us, Haruo Yoshida [3] finds a method to extend the symplectic integrators to any even order of the Zassenhaus formula.

# The Yoshida Symplectic Integrator

We will not prove here the Yoshida technique that is well described in the paper, but we will say that it exploits the time reversal symmetry of the BCH formula. This generates a symmetry in the solution such that if a symmetric integrator of order $2n$, $S_{2n}(t)$, is already known, a $(2n+2)^{\text{th}}$ order integrator is obtained as

$$S_{2n+2}(t) = S_{2n}(z_1 t) S_{2n}(z_0 t) S_{2n}(z_1 t) \tag{31}$$

where $z_0$ and $z_1$ are solutions of

$$z_0 + 2z_1 = 1; \quad z_0^{2n+1} + 2z_1^{2n+1} = 0 \tag{32}$$

or

$$z_0 = -\frac{2^{1/(2n+1)}}{2 - 2^{1/(2n+1)}}; \quad z_1 = \frac{1}{2 - 2^{1/(2n+1)}}. \tag{33}$$

Where $n = 1$ for fourth order, $n = 2$ for sixth order and so on.

# Exercise: fourth order symplectic integrator

Recalling that the second order integrator is

$$S_2(t) = e^{-t\frac{1}{2}:T:}e^{-t:V:}e^{-t\frac{1}{2}:T:} \tag{34}$$

so with coefficients $\left[\frac{1}{2}, 1, \frac{1}{2}\right]$ use the Yoshida method to calculate the fourth order integrator and verify the result with the coefficients (27-30).

## Computational Exercise: the sextupole

Generate a 1D gaussian particle distribution in (x, px) with the following characteristics:

$$N = 1e6 \quad (35)$$

$$\sigma_x = 2.5e-3 \text{ m} \quad (36)$$

$$\sigma_{p_x} = 2.5e-3. \quad (37)$$

Calculate the Twiss functions $\alpha$, $\beta$, $\gamma$ and emittance. Plot the beam distribution in the phase space and the ellipse with equation

$$\gamma x^2 + 2\alpha x p_x + \beta p_x^2 - 4\pi\epsilon = 0. \quad (38)$$

# Computational Exercise: the sextupole

Using the Hamiltonian of a sextupolar magnet given by

$$H = \frac{p_x^2}{2} + k_s \frac{x^3}{6} \tag{39}$$

with length $L = 6$ m and gradient $k_s = 1.0\ \frac{\mathrm{T}}{\mathrm{m}^2}$ transport the particles and the ellipse

- with the Lie transform truncated at the order 10;
- with the Yoshida integrator at the order 4;

for both cases plot the particles and the transported ellipse.

Solutions to proposed exercises.

# Solution: Lie operator iteration of single variable function

If $f(x)$ is a function only of $x$ and $g(p_x)$ is a function only of $p_x$ prove that

$$\begin{aligned}
(:H_x:)^n f(x) &= 0 && (40)\\
(:H_x:)^n g(p_x) &= \left(\frac{\partial H_x}{\partial x}\right)^n \frac{\partial^n g(p_x)}{\partial p_x^n} && (41)\\
(:H_{p_x}:)^n f(x) &= (-1)^n \left(\frac{\partial H_{p_x}}{\partial p_x}\right)^n \frac{\partial^n f(x)}{\partial x^n} && (42)\\
(:H_{p_x}:)^n g(p_x) &= 0 && (43)
\end{aligned}$$

# Solution: Lie operator iteration of single variable function

We start calcualting

$$
\begin{aligned}
:H_x : f(x) &= \frac{\partial H_x}{\partial x}\cancel{\frac{\partial f(x)}{\partial p_x}} - \cancel{\frac{\partial H_x}{\partial p_x}}\frac{\partial f(x)}{\partial x} = 0 && (44)\\
:H_x : g(p_x) &= \frac{\partial H_x}{\partial x}\frac{\partial g(p_x)}{\partial p_x} - \cancel{\frac{\partial H_x}{\partial p_x}}\cancel{\frac{\partial g(p_x)}{\partial x}} && (45)\\
:H_{p_x} : f(x) &= \cancel{\frac{\partial H_{p_x}}{\partial x}}\cancel{\frac{\partial f(x)}{\partial p_x}} - \frac{\partial H_{p_x}}{\partial p_x}\frac{\partial f(x)}{\partial x} && (46)\\
:H_{p_x} : g(p_x) &= \cancel{\frac{\partial H_{p_x}}{\partial x}}\frac{\partial g(p_x)}{\partial p_x} - \frac{\partial H_{p_x}}{\partial p_x}\cancel{\frac{\partial g(p_x)}{\partial x}} = 0 && (47)
\end{aligned}
$$

## Solution: Lie operator iteration of single variable function

The iterations of non-zero equations are

$$\begin{aligned}(:H_x:)^2 g(p_x) &= \frac{\partial H_x}{\partial x}\frac{\partial}{\partial p_x}\left(\frac{\partial H_x}{\partial x}\frac{\partial g(p_x)}{\partial p_x}\right) \\ &= \left(\frac{\partial H_x}{\partial x}\right)^2 \frac{\partial^2 g(p_x)}{\partial p_x^2} \qquad (48)\end{aligned}$$

$$\begin{aligned}(:H_{p_x}:)^2 f(x) &= -\frac{\partial H_{p_x}}{\partial p_x}\frac{\partial}{\partial x}\left(-\frac{\partial H_{p_x}}{\partial p_x}\frac{\partial f(x)}{\partial x}\right) \\ &= \left(\frac{\partial H_{p_x}}{\partial p_x}\right)^2 \frac{\partial^2 f(x)}{\partial x^2} \qquad (49)\end{aligned}$$

## Solution: Lie operator iteration of single variable function

It is then clear that

$$(: H_x :)^n f(x) = 0 \tag{50}$$

$$(: H_x :)^n g(p_x) = \left(\frac{\partial H_x}{\partial x}\right)^n \frac{\partial^n g(p_x)}{\partial p_x^n} \tag{51}$$

$$(: H_{p_x} :)^n f(x) = (-1)^n \left(\frac{\partial H_{p_x}}{\partial p_x}\right)^n \frac{\partial^n f(x)}{\partial x^n} \tag{52}$$

$$(: H_{p_x} :)^n g(p_x) = 0 \tag{53}$$

# Solution: second order integrator

Considering the Zassenhaus formula at the second order

$$e^{-t:T+V:} \approx e^{-t:T:} e^{-t:V:} e^{\frac{t^2}{2}:\{T,V\}:} \tag{54}$$

show that if the Hamiltonian is split as $\frac{1}{2}T + V + \frac{1}{2}T$ the second order cancels and the equation becomes

$$e^{-t:\frac{1}{2}T+V+\frac{1}{2}T:} \approx e^{-t:\frac{1}{2}T:} e^{-t:V:} e^{-t:\frac{1}{2}T:}. \tag{55}$$

Hints: the Poisson Brackets satisfy the following rules

$$\{A+B, C\} = \{A, C\} + \{B, C\} \tag{56}$$
$$\{A, B+C\} = \{A, B\} + \{A, C\} \tag{57}$$
$$\{k_1 A, k_2 B\} = k_1 k_2 \{A, B\} \tag{58}$$
$$\{A, A\} = 0 \tag{59}$$
$$\{A, B\} = -\{B, A\} \tag{60}$$

# Solution: second order integrator

We apply the Zassenhaus formula

$$
\begin{aligned}
e^{-t:\frac{1}{2}T+V+\frac{1}{2}T:} &= e^{-t:\frac{1}{2}T+\left(V+\frac{1}{2}T\right):} \\
&= e^{-t:\frac{1}{2}T:}e^{-t:V+\frac{1}{2}T:}e^{\frac{t^2}{2}:\{\frac{1}{2}T,V+\frac{1}{2}T\}:} \\
&= e^{-t:\frac{1}{2}T:}e^{-t:V:}e^{-t:\frac{1}{2}T:} \\
&\quad e^{\frac{t^2}{2}:\{V,\frac{1}{2}T\}:}e^{\frac{t^2}{2}\left(:\{\frac{1}{2}T,V\}:+:\{\frac{1}{2}T,\frac{1}{2}T\}:\right)} \qquad (61)
\end{aligned}
$$

the properties of the Poisson brackets tell us that $\{\frac{1}{2}T, \frac{1}{2}T\} = 0$ and $\{\frac{1}{2}T, V\} = -\{V, \frac{1}{2}T\}$ so

$$
\begin{aligned}
&e^{\frac{t^2}{2}:\{V,\frac{1}{2}T\}:}e^{\frac{t^2}{2}\left(:\{\frac{1}{2}T,V\}:+:\{\frac{1}{2}T,\frac{1}{2}T\}:\right)} \\
&= e^{\frac{t^2}{2}:\{V,\frac{1}{2}T\}:}e^{-\frac{t^2}{2}:\{V,\frac{1}{2}T\}:} = \mathbb{1}.
\end{aligned}
$$

# Solution: fourth order symplectic integrator

Recalling that the second order integrator is

$$S_2(t) = e^{-t\frac{1}{2}:T:}e^{-t:V:}e^{-t\frac{1}{2}:T:} \tag{62}$$

so with coefficients $[\frac{1}{2}, 1, \frac{1}{2}]$ use the Yoshida method to calculate the fourth order integrator and verify the result with the coefficients (27-30).

# Solution: fourth order symplectic integrator

First of all we calculate the $z_0$ and $z_1$ for order 4, i.e. $n = 1$.

$$z_0 = -\frac{2^{1/(2n+1)}}{2 - 2^{1/(2n+1)}} = -\frac{2^{1/3}}{2 - 2^{1/3}} = -1.702414 \qquad (63)$$

$$z_1 = \frac{1}{2 - 2^{1/(2n+1)}} = \frac{1}{2 - 2^{1/3}} = 1.3512. \qquad (64)$$

Then the $4^{\text{th}}$ order integrator has coefficients

$$z_1 \left[\frac{1}{2}, 1, \frac{1}{2}\right], z_0 \left[\frac{1}{2}, 1, \frac{1}{2}\right], z_1 \left[\frac{1}{2}, 1, \frac{1}{2}\right] \qquad (65)$$

these are 9 coefficients, but the last coefficient of the first series is a drift that is joint with the first coefficient of the second series, so we can sum them

$$\left[\frac{z_1}{2}, z_1, \frac{z_1}{2} + \frac{z_0}{2}, z_0, \frac{z_0}{2} + \frac{z_1}{2}, z_1, \frac{z_1}{2}\right] = \qquad (66)$$

$$[0.6756, 1.3512, -0.1756, -1.702414, -0.1756, 1.3512, 0.6756] .$$

# Solution: the sextupole

Generate a 1D gaussian particle distribution in (x, px) with the following characteristics:

$$\begin{aligned} N &= 1e6 && (67) \\ \sigma_x &= 2.5e-3 \text{ m} && (68) \\ \sigma_{p_x} &= 2.5e-3. && (69) \end{aligned}$$

Calculate the Twiss functions $\alpha$, $\beta$, $\gamma$ and emittance. Plot the beam distribution in the phase space and the ellipse with equation

$$\gamma x^2 + 2\alpha x p_x + \beta p_x^2 - 4\pi\epsilon = 0. \tag{70}$$

## Solution: the sextupole

Using the Hamiltonian of a sextupolar magnet given by

$$H = \frac{p_x^2}{2} + k_s \frac{x^3}{6} \tag{71}$$

with length $L = 6$ m and gradient $k_s = 1.0\ \frac{\mathrm{T}}{\mathrm{m}^2}$ transport the particles and the ellipse

- with the Lie transform truncated at the order 10;
- with the Yoshida integrator at the order 4;

for both cases plot the particles and the transported ellipse.

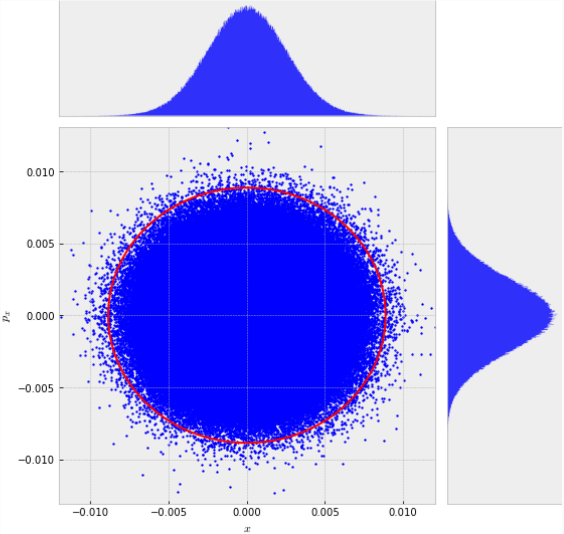
# Solution: the sextupole

Define beam, Twiss functions and ellipse.

```
sx = 2.5e-3
sp = 2.5e-3
N = 1000000
xi = numpy.random.normal(0,sx,N)
pi = numpy.random.normal(0,sp,N)
sx=(xi.std())**2
sp=(pi.std())**2
x_lim = max(abs(xi.min()), xi.max())
p_lim = max(abs(pi.min()), pi.max())
sxp=((xi-xi.mean())*(pi-pi.mean())).sum()/N
emit = numpy.sqrt(sx*sp-sxp**2)
alpha = -sxp/emit
beta = sx/emit
gamma = sp/emit
x_ellipse = numpy.linspace(-x_lim, x_lim, 1000)
p_ellipse = numpy.linspace(-p_lim, p_lim, 1000)
x_ellipse, p_ellipse = numpy.meshgrid(x_ellipse, p_ellipse)
ellipse = gamma*x_ellipse**2+2*alpha*x_ellipse*p_ellipse+
    beta*p_ellipse**2-4*numpy.pi*emit
```

# Solution: the sextupole

This is how the beam and the ellipse appear.



# Solution: the sextupole

Create the Lie transform function.

```
x, p = sympy.symbols('x,p_x', real=True)
def lie_operator(f, g):
    return f.diff(x)*g.diff(p)-f.diff(p)*g.diff(x)
def lie_transform(f, g, order):
    step = lie_operator(f, g)
    result = g + step
    for i in range(2,order+1,1):
        step = sympy.simplify(lie_operator(f, step))
        result = sympy.simplify(result + step/sympy.factorial(i))
    return result
```

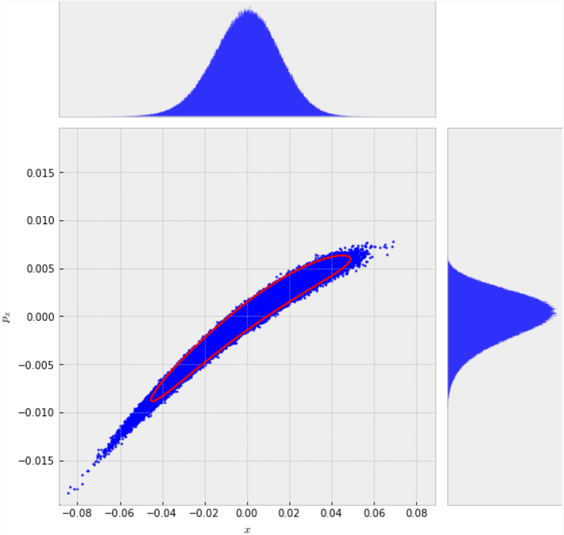
# Solution: the sextupole

Define the Hamiltonian and track the particles and the ellipse.

```
L = 6
ks = 1.0
H = (p**2/2+ks*x**3/6)
order=10
xf = sympy.lambdify((x,p),lie_transform(-L*H,x,order))
pf = sympy.lambdify((x,p),lie_transform(-L*H,p,order))
x_final_trunc = xf(xi,pi)
p_final_trunc = pf(xi,pi)
x_lim = max(abs(x_final_trunc.min()), x_final_trunc.max())
p_lim = max(abs(p_final_trunc.min()), p_final_trunc.max())
x_ellipse = numpy.linspace(-x_lim, x_lim, 1000)
p_ellipse = numpy.linspace(-p_lim, p_lim, 1000)
x_ellipse, p_ellipse = numpy.meshgrid(x_ellipse, p_ellipse)
ellipse_sym = sympy.lambdify((x,p), lie_transform(L*H,gamma*
    x**2+2*alpha*x*p+beta*p**2-4*numpy.pi*emit,order), numpy
    )
ellipse=ellipse_sym(x_ellipse, p_ellipse)
```

# Solution: the sextupole

This is how the beam and the ellipse appear after the transport.



# Solution: the sextupole

Create the Lie transform function for exact solution

```python
def lie_operator(f, g):
    return f.diff(x)*g.diff(p)-f.diff(p)*g.diff(x)
def lie_transform(f, g):
    step = lie_operator(f, g)
    result = g + step
    order = 2
    while step != 0:
        step = sympy.simplify(lie_operator(f, step))
        result = sympy.simplify(result + step/sympy.factorial(order))
        order = order + 1
    return result
```

# Solution: the sextupole

Create Yoshida coefficients calculator.

```
def symplectic_integrator_numeric(order):
    if order % 2 != 0:
        print "This integrator is only for even order"
        return
    S = [0.5, 1, 0.5]
    if order > 2:
        for n in range(1,int(order/2)):
            alpha = 2.0**(1.0/(2.0*n+1))
            x0=-alpha/(2.0-alpha)
            x1=1/(2.0-alpha)
            TC=[i*x0 for i in S]
            TL=[i*x1 for i in S]
            T=[]
            for i in TL[:-1]:
                T.append(i)
            T.append(TL[-1]+TC[0])
            for i in TC[1:-1]:
                T.append(i)
            T.append(TC[-1]+TL[0])
            for i in TL[1:]:
                T.append(i)
            S=T
    return S
```

# Solution: the sextupole

Prepare the two half Hamiltonians, calculate the coefficients and set the intial particles.

```
Hd = p**2/2
Hk = ks*x**3/6
order = 4
coeff = symplectic_integrator_numeric(order)
xinit = xi
pinit = pi
counter = 0
```

# Solution: the sextupole

To the Lie transform for each coefficient of the integrator.

```
for i in coeff:
    if counter%2 == 0:
        Hd_numeric_x = sympy.lambdify((x,p), lie_transform(-L*i*Hd, x), numpy)
        Hd_numeric_p = sympy.lambdify((x,p), lie_transform(-L*i*Hd, p), numpy)
        ellipse = lie_transform(L*i*Hd, ellipse)
        xf = Hd_numeric_x(xinit, pinit)
        pf = Hd_numeric_p(xinit, pinit)
    else:
        Hk_numeric_x = sympy.lambdify((x,p), lie_transform(-L*i*Hk, x), numpy)
        Hk_numeric_p = sympy.lambdify((x,p), lie_transform(-L*i*Hk, p), numpy)
        ellipse = lie_transform(L*i*Hk, ellipse)
        xf = Hk_numeric_x(xinit, pinit)
        pf = Hk_numeric_p(xinit, pinit)

    xinit = xf
    pinit = pf
    counter = counter + 1
```

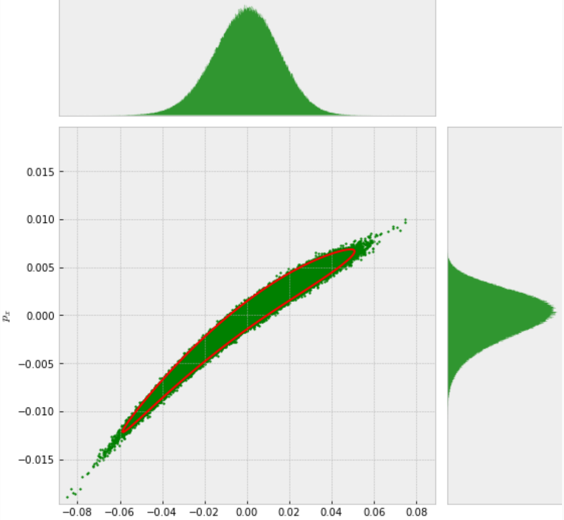
# Solution: the sextupole

Finalize the ellipse.

```
x_final_symp = xf
p_final_symp = pf
x_lim = max(abs(x_final_symp.min()), x_final_symp.max())
p_lim = max(abs(p_final_symp.min()), p_final_symp.max())
x_ellipse = numpy.linspace(-x_lim, x_lim, 1000)
p_ellipse = numpy.linspace(-p_lim, p_lim, 1000)
x_ellipse, p_ellipse = numpy.meshgrid(x_ellipse, p_ellipse)
ellipse_sym = sympy.lambdify((x,p), ellipse, numpy)
ellipse=ellipse_sym(x_ellipse, p_ellipse)
```

# Solution: the sextupole

This is how the beam and the ellipse appear after the transport with the symplectic integrator.



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