1. Kinematics, cross-sections etc

A study of kinematics is of great importance to any experiment on particle scattering. It is necessary to interpret your measurements, but at an earlier stage to determine where to put your detectors so that the particles will actually pass through them.
The word kinematics is used to describe the relations which follow from the conservation of energy and momentum. For the most part in this course we will be dealing with fast particles and *relativistic* kinematics. Before looking at how to apply energy-momentum conservation in different situations, we need to remind ourselves of the fundamentals: the Lorentz transformation relating observables in different reference frames; and the relationships between different quantities such as energy, mass and velocity for a high-energy particle in a given reference frame. We describe the interactions of high-energy particles in terms of the exchange of force particles, such as the photon. These fleeting *virtual particles* appear not to obey all the rules. However longer-lived *real* particles do obey the conservation laws and they are the ones you actually detect after scattering processes. They **are** 'on the mass shell'.

1.1 Basics - 4-vectors and Lorentz transformations

In introductory relativity the formalism is usually developed by reference to *intervals* in space and time in different frames. In the study of particle interactions we are (almost) exclusively concerned with the application to particle energy and momentum. Using the concept of a *4-vector*, we can relate the transformation equations as applied to energy and momentum to those for space and time, or indeed other quantities such as charge and current densities which will not concern us here. A 4-vector consists of a "time" or scalar component and a "space" or (3-)vector component. These components behave like scalars and vectors, respectively, when the coordinate system is *rotated.* However they are mixed by a Lorentz transformation or *boost* between frames moving with a constant relative velocity.

A note on **units:** we will consistently use particle momenta and masses measured in energy units, usually MeV or GeV. Thus there are no factors of *c* in any of our relativistic relationships. This not only tidies up the algebra, but also introduces a symmetry into the Lorentz transformation equations. In the case of the space-time relations, however, it is usual and convenient to measure space and time in different units - space intervals in metres and time in second, or multiples thereof. We therefore retain the factor of *c* here in such a way that all components of the space-time 4-vector are expressed as distances.

Momentum and energy form the four components of a *4-vector* (E, p_x, p_y, p_z) which behaves much like the space-time 4-vector (ct, x, y, z). The transformations of energy and momentum are then, taking the z direction along the transformation:

$$
\begin{aligned}
E' &= \gamma(E - \beta p_z) \\
p_x' &= p_x \\
p_y' &= p_y \\
p_z' &= \gamma(p_z - \beta E)
\end{aligned}
$$

where

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}\ .$$

The components of any other 4-vector transform in an analogous way.

It is frequently the case that we are interested only in transformations along the direction of the particle's momentum. Examples include the transformation between the LAB and CM frames for a pair of interacting particles, or between a particle's *rest frame* and some moving frame. In this case we do not need to worry about the components of *p*, and the equations take the simple, symmetric form:

$$E' = \gamma(E-\beta p)$$
$$p' = \gamma(p-\beta E)$$

It is a general property of the Lorentz Transformation that it leaves some quantity *invariant* (just as a rotation in 3 dimensions leaves the length of a 3-vector quantity unchanged). In the case of the energy-momentum transformation the quantity E^2-p^2 has the same value in all reference frames (as can be checked explicitly using the transformation equations).

1.2 The relativistic energy-momentum-mass relations

1.2.1 Energy and momentum in rest and moving frames

Suppose (for the moment) that a particle has energy E_0 in its *rest frame*, where its velocity and momentum are zero. In another frame where its velocity is $v=\beta c$, the Lorentz transformation gives

$$E'=\gamma(E_0-\ 0)$$
$$p'=\gamma(0-\beta E_0)$$

or

$$E'=\gamma E_0$$
$$p'=\beta\gamma E_0$$

Of course the rest frame energy E_0 is equivalent to the particle *mass m*, and we can see that the Lorentz invariant quantity E^2-p^2 is equal to the square of this energy. Thus relativity gives the following expressions relating the various kinematic quantities:

$$E = \gamma m$$
$$p = \beta\gamma m$$
$$\beta = {p}/{E}$$
$$E^2 = p^2 + m^2$$
$$E = \gamma m$$

The high-energy or *relativistic limit* of these relationships gives E=p, β=1. For this reason it is common to use the energy and momentum of a particle interchangeably.

1.2.2 Time dilation and mean decay distance

It is relevant here to derive an expression for the *decay distance* of a moving particle. If the particle lives for a time τ in its rest frame, in a moving frame this becomes γτ due to time dilation. During this time it travels a distance $v\gamma\tau$. Using v=βc, and substituting from the above relations, we can see that

$$\text{Decay distance} = \beta\gamma c\tau = p\frac{c\tau}{m}$$

i.e., the mean decay distance of a particle is proportional to its *momentum.*

1.3 Problems in relativistic kinematics

Some important concepts in particle physics:

- The **Centre-of-Mass** frame for a system of interacting particles is that in which the total momentum summed over all particles is zero. The system may be a beam particle and stationary target, a pair of colliding beams, the final state particles produced in an interaction or some subset of these. The Centre-of-Mass frame is also called the Centre-of-Momentum or CM frame. In a sense it is analogous for the system of particles, seen as a whole, to the rest frame for a single particle.

- The **laboratory** or LAB frame is that in which our experimental apparatus is at rest! In different experiments it may be the rest frame of one of the initial state particles (fixed target), or the same as the CM frame for the initial particles - or neither.

- The **invariant mass** of a system of particles is the system's total energy in its CM frame. It may be found from the total energy and momentum in *any* frame:

$$(\text{Total Energy})^2 = (\text{Total Momentum})^2 + (\text{Invariant Mass})^2$$

- The **total CM energy** in an experiment is the invariant mass of all initial (or final) state particles. The symbol s is used for the *square* of the total CM energy. From the previous bullet we can see that s is given by:

$$s = (\text{Total Energy})^2 - (\text{Total Momentum})^2$$

- The **4-momentum transfer** in an interaction is the energy-momentum *change* of one of the particles between initial and final states. It may be thought of as the energy and momentum carried by the exchanged *virtual* particle. These quantities vary according to which frame we are working in, but there is a quantity analogous to the mass of the virtual particle which is invariant under changes between frames. This quantity is usually denoted by the symbol q^2:

$$q^2 = (\text{Energy change})^2 - (\text{Momentum change})^2$$

Examples of problems:

The solution of problems frequently involves the calculation of Lorentz invariant quantities, such as q^2 or s. Conservation of energy and momentum are used to give relationships between initial and final state quantities, and the relationships between energy, momentum and mass for the individual particles applied to simplify the results.

We illustrate the techniques used by means of a series of examples. The following useful relationships can be derived for different experimental situations:

- In a fixed-target experiment, where beam particles of rest mass m_B and energy E_B are incident on stationary target particles of mass m_T, the total CM frame energy is $\sqrt{s}$, where $s = m_B^2 + m_T^2 + 2m_T E_B$. The last term in this expression is the dominant one for large E_B. In such an experiment, if it is required to have CM energy larger than a certain value M, this translates into a requirement that E_B be larger than

$$\frac{M^2 - m_B{}^2 - m_T{}^2}{2m_T}.$$

- If a short-lived particle is produced in an interaction and its decay products are measured, the mass of the short-lived particle is equal to the invariant mass of the system of decay products. In the case of a two-body decay to particles with energies E_B, E_C much larger than their masses, the invariant mass is

$$2E_B E_C (1 - \cos\theta_{\mathrm{BC}}).$$

- In a scattering experiment, where the energy and angle of deflection of the outgoing scattered particle are measured, the q^2 of the scattering is

$$2m^2 - 2EE' + 2pp'\cos\theta.$$

Here m is the mass of the scattered particle, E, E' its energy before and after the scattering and p, p' the corresponding momenta, with θ the angle of scatter. In the high-energy limit the expression simplifies to

$$q^2 = -4EE'\sin^2 \theta/2 \, .$$

- For elastic scattering we can use energy and momentum conservation to relate E' and the angle θ. This allows us to write q^2 in terms of E' alone:

$$q^2 = -2m_T\left(E - E'\right)$$

where m_T is the target particle mass.

1.4 Cross Sections and decay rates

In most cases we are concerned with the measurement of individual events. However it is important to remember that our ultimate aim in performing an experiment, whether in particle physics or any of the other applications of particle detection techniques, is to collect large numbers of events. The important information will come from measuring the rates of different types of event. In this section we discuss briefly the two most important quantities that are measured in particle physics experiments: the cross section for a particular interaction, and the decay rate for an unstable particle.

The rate at which interactions occur will be proportional to some factors dependent on the experimental setup, such as beam rates and target densities, and to a quantity known as the *cross section* σ. The cross section for a particular reaction is the fundamental measurable quantity which incorporates all effects such as the strength of the underlying interaction, propagator factors for virtual exchange particles, and dependence on the available energy, or phase space.

It is called a cross section because it has the units of area. It may be thought of as a small effective area centred on the target such that if the incident particle should pass through this area then the reaction would occur. This picture is not physically realistic, but it gives the correct rates. If we think of the cross section as an area perpendicular to the direction in which the particles are moving, then this area will not be affected by a Lorentz transformation along the direction of the incident particles and the cross section will be the same in the LAB and CM frames. Cross sections are commonly given in a unit called a *barn* (10^{-28} m^2), or multiples thereof such as *nanobarns nb* or *picobarns pb*.

There are different expressions relating the cross section to the observed interaction rate W, depending on the experimental setup. In a fixed target experiment, if the target area is larger than the beam so that all beam particles pass through the target, we can write

$$W = r \cdot \rho l \cdot \sigma$$

where r is the beam particle rate, ρ the number density of target particles per unit volume and l the length of the target, so that ρl gives the number of target particles per unit area.

If, on the other hand, the beam is fairly wide with a flux *J* particles per unit area per second, the expression is

$$W = J \cdot n \cdot \sigma$$

with *n* the total number of target particles.

In a colliding beam setup, things are more complicated. The particles in each beam are stored in “bunches”, which arrive at the collision point with a certain frequency. The interaction rate will depend on the numbers of particles in the bunches, the collision frequency and the size of the bunches - the smaller the bunch, the higher the rate, since the particles in a small bunch are packed more closely together. Some expressions are

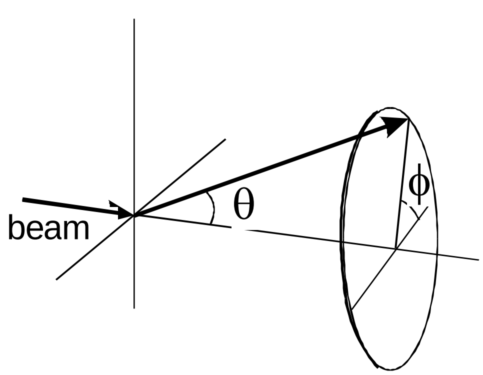
$$\begin{aligned} W &= \frac{n_1 n_2 f}{A}\sigma \\ &= \frac{I_1 I_2}{Af}\sigma \end{aligned}$$

where $n_{1,2}$ are the numbers of particles per beam, *f* the collision frequency and *A* is some effective overlap area for the beams. In the second line this is re-written in terms of the beam *currents* $I=nf$, i.e. number of particles per unit time in each beam.

The product of these factors for a colliding beam machine is usually referred to as the *luminosity*. It can be thought of as the “brightness” of the source of particle collision processes. It has units $(\text{area}\times\text{time})^{-1}$, so that when multiplied by the cross section the result is a rate of interactions. Clearly the luminosity can be defined also for the fixed target configurations discussed above, although this is less usual.

1.4.1 Differential cross-sections

The quantity defined above is the *total* cross section σ for an interaction. If you are to get as much information as possible from a scattering process you can observe, not only that an interaction occurred, but also how the resultant scattered or transformed particles come out of the target afterwards.



The cross section σ may be written as a sum of contributions for different directions of emission of the final particles. For two-body final states it is only necessary to give the direction for one particle as the conservation of energy and momentum then determines the direction for the other. The particle direction is defined by the polar angle θ and the azimuthal angle ϕ in a set of axes such that *z* is along the beam.

This gives the *differential cross-section*

$$\frac{d\sigma(\theta,\phi)}{d\Omega}$$

which is defined by

$$dW = JN\frac{d\sigma(\theta,\phi)}{d\Omega}d\Omega$$

where dW is the rate for particles emitted into the *solid angle* $d\Omega$ (= $d(\cos\theta)d\phi$) in the direction given by θ and ϕ.

$$\sigma = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \frac{d\sigma(\theta,\phi)}{d\Omega}$$

The reaction cross-section σ is found by integrating the differential cross section over all solid angles.

Often the beams and targets are *unpolarised* and any dependence on the azimuthal angle ϕ averages out. In this case the differential cross section is a function of θ only.

If the interaction rate is measured as a function of other variables such as the outgoing particle energies, other types of differential cross section may be measured. It is possible also to generalise the idea and measure multiply differential cross sections, e.g.

$$\frac{d^2\sigma}{dE_1 dE_2}$$

as a function of the energies of 2 final state particles.

1.4.2 Particle Decays

The rate of decay of a short-lived particle can also yield valuable information about particle properties, such as the strength of the interaction by which the decay proceeds. The important quantity here is the decay rate $W=1/\tau$ in the particle rest frame. τ here is the mean lifetime, which for a moving particle is extended by time dilation, as discussed above. For very short-lived particles, a related quantity is the *width* Γ, or uncertainty in the mass, given by $\Gamma=\eta W$. For a particle with several different modes of decay, one can also define *partial* decay rates W_i or widths Γ_i for individual modes. The total width is the sum of all partial widths.