Momentum eigenstates

- linear momentum eigenstates
- orbital angular momentum eigenstates
- spin angular momentum eigenstates
- addition of angular momenta

The generic angular momentum operator

We will work mainly with quantum numbers in the applications
Linear momentum eigenstates

We will consider motion in one dimension (x):

\[ p_x = -i \frac{\hbar}{2\pi} \frac{d}{dx} \]

By solving the eigenvalue equation \( p_x \psi = p_x \psi \)
(where both \( \psi \) and \( p_x \) are unknown), we find

\[ \psi(x) = c \cdot e^{\left( \frac{2\pi i}{\hbar} \right)xp_x} \]
Eigenstates of orbital angular momentum

We will look for the simultaneous eigenstates $\psi$ of the operators $L^2$ and $L_z$

$$L_z \psi = L_z \psi, \quad L^2 \psi = L^2 \psi$$

We will use spherical coordinates: $(r, \theta, \phi)$
$L_z$ eigenstates and eigenvalues

The eigenvalue equation
\[-i \frac{h}{2\pi} \frac{d\psi}{d\phi} = L_z\]
gives
\[\psi(\phi) = ce \left( \frac{2\pi i}{h} \right) \phi L_z\]
\[\psi(\phi) = \psi(\phi + 2\pi) \Rightarrow L_z = m \frac{h}{2\pi}\]
\[m = 0, \pm 1, \pm 2, \ldots\] is the magnetic quantum number
Eigenstates of $L^2$ and $L_z$

Spherical harmonics

$$Y_{lm}(\theta, \phi) = (-1)^m \left[ \frac{(2l + 1)(l - m)!}{4\pi(l + m)!} \right]^{1/2} P^m_l(\cos \theta) e^{im\phi}$$

where $P^m_l(\cos \theta)$ are the associated Legendre functions

$$L^2 = \left( \frac{\hbar}{2\pi} \right)^2 l(l + 1), \quad l = 0, 1, 2, \ldots$$

the orbital quantum number

the magnetic quantum number is restricted by $l$:

$$m = -l, -l + 1, \ldots, 0, \ldots, l - 1, l$$
Parity eigenstates

The spherical harmonics are also eigenstates of the parity operator

\[ PY_{lm} = (-1)^l Y_{lm} \]

Application: transitions in atoms
The generic angular momentum operator

\[ J = J_x \mathbf{\hat{u}}_x + J_y \mathbf{\hat{u}}_y + J_z \mathbf{\hat{u}}_z, \quad [J_x, J_y] = i \frac{\hbar}{2\pi} J_z \] and cyclic permutations

\[ J^2 = \left( \frac{\hbar}{2\pi} \right)^2 j(j + 1), \quad L_z = m \frac{\hbar}{2\pi}, \quad j = 0, 1, 2, \ldots, \quad m_j = -j, -j + 1, \ldots, j \]

By solving the eigenvalue equations for \( J^2 \) and \( J_z \) simultaneously and assuming \( n \) eigenstates, we find that there are two ‘types’ of angular momentum:

- integer \( j \)
- half-integer \( j \)
The spin angular momentum

Spin does not involve a rotation in space

\[ S^2 = \left( \frac{\hbar}{2\pi} \right)^2 s(s + 1) \]

where \( s \) is the spin quantum number

\[ S_z = m_s \frac{\hbar}{2\pi}, \quad m_s = -s, -s + 1, \ldots, s - 1, s \]

✓ integer \( s \): bosons
✓ half-integer \( s \): fermions
Addition of angular momenta

Let us assume that we want to ‘add’ the quantum numbers $l$ and $s$ to find the quantum number of the total angular momentum, $j$, and the quantum number of the z-component, $m_j$.

The rules are:

1. $j = |l - s|, \ldots, l + s$
2. for each $j$, we have $2j + 1$ $m_j$’s: $m_j = -j, -j + 1, \ldots, j$

Application: L-S coupling and J-J coupling in atoms