Momentum eigenstates

- linear momentum eigenstates
- orbital angular momentum eigenstates
- spin angular momentum eigenstates
- addition of angular momenta

- the generic angular momentum operator

We will work mainly with quantum numbers in the applications

Linear momentum eigenstates

We will consider motion in one dimension (x):

$$\boldsymbol{p_x} = -i\frac{h}{2\pi}\frac{\mathrm{d}}{\mathrm{d}x}$$

By solving the eigenvalue equation $\boldsymbol{p_x}\psi = p_x\psi$

(where both ψ and p_x are unknown), we find

$$\psi(x) = c \cdot e^{\left(\frac{2\pi i}{h}\right)xp_x}$$

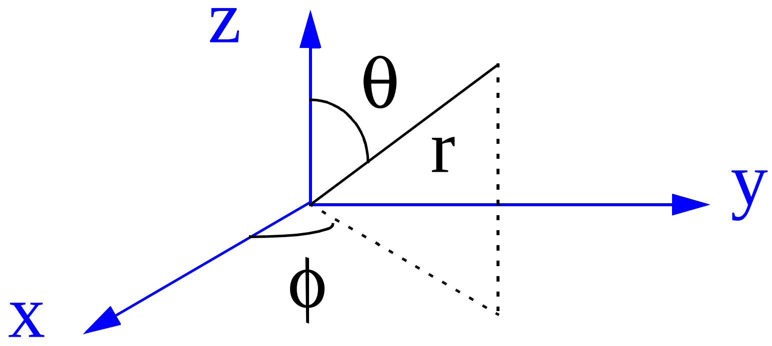
Eigenstates of orbital angular momentum

We will look for the simultaneous eigenstates ψ

of the operators $\boldsymbol{L}^2$ and $\boldsymbol{L_z}$

$$\boldsymbol{L_z}\psi = L_z\psi, \quad \boldsymbol{L}^2\psi = L^2\psi$$

We will use spherical coordinates: (r,θ,ϕ)



L_z eigenstates and eigenvalues

The eigenvalue equation

$$-i\frac{h}{2\pi}\frac{d\psi}{d\phi} = L_z$$ gives

$$\psi(\phi) = ce^{\left(\frac{2\pi i}{h}\right)\phi L_z}$$

Condition: $\psi(\phi) = \psi(\phi + 2\pi) \Rightarrow L_z = m\frac{h}{2\pi}$ **quantized**

$m = 0, \pm 1, \pm 2, \ldots$ is the magnetic quantum number

Eigenstates of L^2 and L_z

Spherical harmonics

$$Y_{lm}(\theta, \phi) = (-1)^m \left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{1/2} P_l^m(\cos\theta) e^{im\phi}$$

where $P_l^m(\cos\theta)$ are the associated Legendre functions

$L^2 = \left(\frac{h}{2\pi}\right)^2 l(l+1), \ l = 0, 1, 2, \ldots$ the orbital quantum number

the magnetic quantum number is restricted by l:

$$m = -l, -l+1, \ldots, 0, \ldots, l-1, l$$

Parity eigenstates

The spherical harmonics are also eigenstates
of the parity operator

$$\boldsymbol{P} Y_{lm} = (-1)^l Y_{lm}$$

Application: transitions in atoms

The generic angular momentum operator

$\boldsymbol{J} = \boldsymbol{J_x}\vec{u}_x + \boldsymbol{J_y}\vec{u}_y + \boldsymbol{J_z}\vec{u}_z$, $[\boldsymbol{J_x}, \boldsymbol{J_y}] = i\frac{h}{2\pi}\boldsymbol{J_z}$ and cyclic permutations

$J^2 = \left(\frac{h}{2\pi}\right)^2 j(j+1)$, $L_z = m\frac{h}{2\pi}$, $j = 0, 1, 2, \ldots$, $m_j = -j, -j+1, \ldots, j$

By solving the eigenvalue equations for $\boldsymbol{J}^2$ and $\boldsymbol{J_z}$ simultaneously and assuming n eigenstates, we find that there are two ‘types’ of angular momentum:

- integer j
- half-integer j

The spin angular momentum

Spin does not involve a rotation in space

$$S^2 = \left(\frac{h}{2\pi}\right)^2 s(s+1)$$

where s is the spin quantum number

$$S_z = m_s \frac{h}{2\pi}, \qquad m_s = -s, -s+1, \ldots, s-1, s$$

- integer s: bosons
- half-integer s: fermions

Addition of angular momenta

Let us assume that we want to 'add' the quantum numbers l and s to find the quantum number of the total angular momentum, j, and the quantum number of the z-component, m_j

The rules are:

1. $j = |l - s|, \ldots, l + s$
2. for each j, we have $2j + 1$ m_j's: $m_j = -j, -j + 1, \ldots, j$

Application: L-S coupling and J-J coupling in atoms