Energy eigenstates

We look for the states of constant energy, $\Psi(x, y, z, t)$

$$i\frac{h}{2\pi}\frac{\partial}{\partial t}\Psi(x, y, z, t) = E\Psi(x, y, z, t)$$

$$\left(-\frac{1}{2m}\left(\frac{h}{2\pi}\right)^2\nabla^2 + V(x, y, z)\right)\Psi(x, y, z, t) = E\Psi(x, y, z, t)$$

We can separate the time- and space-dependence of the wavefunction

$$\Psi(x, y, z, t) = \psi(x, y, z)\phi(t)$$

Energy eigenstates (cont'd)

$$i\frac{h}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = E\phi(t)$$

$$-\frac{1}{2m}\left(\frac{h}{2\pi}\right)^2\nabla^2\psi(x, y, z) + V\psi(x, y, z) = E\psi(x, y, z)$$

(time-independent) Schrödinger equation

The stationary states (energy eigenstates) are

$$\Psi(x, y, z, t) = \psi(x, y, z)e^{-2\pi iEt/h}$$

<u>**Properties of the stationary state**</u>

1. there is always a ground state
2. there may be excited states
3. some energy eigenstates can be degenerate
4. discrete eigenvalues = finite motion = bound state
5. constant probability distribution $|\Psi|^2$
6. constant expectation values

Terminology

energy levels = energy eigenvalues

energy spectrum = the set of energy eigenstates

Application: The free particle

We want to find the stationary states of a particles that moves in the x-direction, V=0, E=K

The Schrödinger equation $-\frac{1}{2m}\left(\frac{h}{2\pi}\right)^2\frac{d^2}{dx^2}\psi = E\psi$

has the solution: $\psi(x) = c_1 e^{ikx} + c_2 e^{-ikx}$

where $k = 2\pi\sqrt{2mE}/h$

Application: The free particle (cont'd)

We include the time-dependence

$$\Psi(x, t) = (c_1 e^{ikx} + c_2 e^{-ikx}) e^{-2\pi i E t / h}$$

non-relativistic motion: $E = p^2 / (2m)$

Combining with $k = 2\pi\sqrt{2mE}/h$, we obtain

$$\Psi(x, t) = (c_1 e^{2\pi i p x / h} + c_2 e^{-2\pi i p x / h}) e^{-2\pi i E t / h}$$

Interpretation of free particle wavefunction

1. superposition of a particle's momentum eigenstates

indeterminate: direction of motion, position

degenerate energy eigenvalues $E = p^2/(2m)$

2. superposition of two plane waves

$$\Psi_1(x, t) = c_1 e^{ikx} e^{-i\omega t} \text{ and } \Psi_2(x, t) = c_2 e^{-ikx} e^{-i\omega t}$$

of angular frequency $\omega = 2\pi f = 2\pi E/h \Rightarrow E = hf$

and wavelength $\lambda = 2\pi/k \Rightarrow \lambda = h/p$

The probability flux

From the time-dependent Schrödinger equation

$$i\frac{h}{2\pi}\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\left(\frac{h}{2\pi}\right)^2\tilde{N}^2\psi + V\psi$$

we find

$$\frac{\partial}{\partial t}|\psi|^2 + \nabla\vec{j} = 0$$ the continuity equation for probability

$$\vec{j} = \frac{h}{4\pi m i}(\psi^*\nabla\psi - \psi\nabla\psi^*)$$ is the probability flux

Properties of the Schrödinger equation

1. ψ is single-valued

2. ψ and $d\psi/dx$ are continuous

3. objects cannot penetrate regions of infinite potential

4. only negative potentials that vanish at infinity can give bound states (application: Coulomb potential)

Matrix elements

They are used to

1. represent operators (and find their eigenstates)
2. study transitions between energy eigenstates

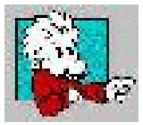
Definition:

$$f_{nm}(t) = \int \Psi_n^* \boldsymbol{f} \Psi_m dV$$

is the matrix element which corresponds to the transition from the stationary state m to the stationary state n

Transitions

Why do we study transitions?



Because that's when radiation is produced

How about $\Delta E \cdot \Delta t = h/(4\pi)$?

Quasi-stationary states

Transition matrix element $V'_{fi} = \int \Psi_f^* V' \Psi_i dV$

The quasi-stationary state has a width Γ and a lifetime

$$\tau = \frac{h}{4\pi\Gamma}$$

The transition (or decay) probability is $\lambda = \frac{1}{\tau}$

Fermi's Golden Rule: $\lambda = \frac{4\pi^2}{h}\left|V'_{fi}\right|^2 \rho(E_f)$